Exam 4 Math 2306 sec. 54

Fall 2021

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: _____

Problem	Points
1	
2	
3	
4	
5	
EC	
Total	

INSTRUCTIONS: There are 5 problems worth 20 points each plus one extra credit problem worth 10 points. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.
No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.

Show all of your work on the paper provided to receive full credit.

1. Suppose y solves the given initial value problem. $\dfrac{dy}{dt} + 3y = 4e^t, \quad y(0) = 2$

$\mathcal{L}\{y' + 3y\} = \mathcal{L}\{4e^t\}$

$sY - y(0) + 3Y = \dfrac{4}{s-1}$

$(s+3)Y - 2 = \dfrac{4}{s-1}$

$(s+3)Y = \dfrac{4}{s-1} + 2$

$Y = \dfrac{4}{(s-1)(s+3)} + \dfrac{2}{s+3}$

Then the Laplace transform of y is (circle the correct $Y(s)$)

(a) $Y(s) = \dfrac{3}{(s-1)(s+2)} + \dfrac{4}{s+2}$

(b) $Y(s) = \dfrac{4}{(s-1)(s+3)} + \dfrac{2}{s+3}$

(c) $Y(s) = \dfrac{4}{(s+1)(s-3)} + \dfrac{2}{s-3}$

(d) $Y(s) = \dfrac{3}{(s+1)(s-2)} + \dfrac{4}{s-2}$

2. Solve the IVP using the Laplace transform.

$$y'' - 2y' + 10y = 0, \quad y(0) = 1, \quad y'(0) = 4$$

Let $Y = \mathcal{L}\{y\}$

$$\mathcal{L}\{y'' - 2y' + 10y\} = \mathcal{L}\{0\} = 0$$

$$s^2 Y - s\underset{1}{y(0)} - \underset{4}{y'(0)} - 2\left(sY - \underset{1}{y(0)}\right) + 10Y(s) = 0$$

$$(s^2 - 2s + 10)Y(s) - s - 4 + 2 = 0$$

$$(s^2 - 2s + 10)Y = s + 2 \Rightarrow Y(s) = \frac{s+2}{s^2 - 2s + 10}$$

$b^2 - 4ac = 4 - 4 \cdot 1 \cdot 10 < 0 \Rightarrow$ Complete the square.

$$s^2 - 2s + 1 - 1 + 10 = (s-1)^2 + 9 = (s-1)^2 + 3^2$$

$$s + 2 = s - 1 + 3$$

$$Y(s) = \frac{s+2}{(s-1)^2 + 3^2} = \frac{s-1}{(s-1)^2 + 3^2} + \frac{3}{(s-1)^2 + 3^2}$$

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2 + 3^2}\right\} = e^{1t}\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 3^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{3}{(s-1)^2 + 3^2}\right\} = e^{1t}\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 3^2}\right\}$$

$$\boxed{y(t) = e^t \cos(3t) + e^t \sin(3t)}$$

3. An RC series circuit with resistance 100 ohms and capacitance 1 millifarad ($\frac{1}{1000}$ farads) has initial charge on the capacitor $q(0) = 1$ Coulomb. After 1 second, a unit impulse $E(t) = \delta(t-1)$ volts is applied. Use the Laplace transform to find the charge $q(t)$ on the capacitor for $t > 0$. That is, solve the IVP

$$100\frac{dq}{dt} + 1000q = \delta(t-1), \quad q(0) = 1.$$

Recall that for any $a \geq 0$, $\mathscr{L}\{\delta(t-a)\} = e^{-as}$.

Let $Q(s) = \mathcal{L}\{q(t)\}$ Put in standard form

$$q' + 10q = \frac{1}{100}\delta(t-1)$$

$$\mathcal{L}\{q' + 10q\} = \mathcal{L}\left\{\frac{1}{100}\delta(t-1)\right\} = \frac{1}{100}e^{-1s}$$

$$sQ(s) - q(0) + 10Q(s) = \frac{1}{100}e^{-s}$$

$$1$$

$$(s+10)Q(s) - 1 = \frac{1}{100}e^{-s}$$

$$Q(s) = \frac{1}{100}\cdot\frac{e^{-s}}{s+10} + \frac{1}{s+10}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\} = e^{-10t}$$

$$\mathcal{L}^{-1}\{e^{-s}F(s)\} = f(t-1)\mathcal{U}(t-1)$$

$$q(t) = \mathcal{L}^{-1}\{Q(s)\}.$$

$$\boxed{q(t) = \frac{1}{100}e^{-10(t-1)}\mathcal{U}(t-1) + e^{-10t}}$$

4. The following integrals formulas may be helpful. For any nonzero constant k,

$$\int x\cos(kx)\,dx = \frac{x}{k}\sin(kx)+\frac{1}{k^2}\cos(kx)+C, \qquad \int x\sin(kx)\,dx = -\frac{x}{k}\cos(kx)+\frac{1}{k^2}\sin(kx)+C.$$

Let $f(x) = 2\pi x$, for $0 < x < \frac{\pi}{2}$. Determine the half-range sine series of f.

$$P = \frac{\pi}{2}, \quad \frac{n\pi x}{P} = \frac{n\pi x}{\frac{\pi}{2}} = 2nx$$

$$b_n = \frac{2}{\pi/2}\int_0^{\pi/2} f(x)\sin(2nx)\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2} (2\pi x)\sin(2nx)\,dx$$

$$= 8\int_0^{\pi/2} x\sin(2nx)\,dx \qquad k = 2n,$$

$$= 8\left[\frac{-x}{2n}\cos(2nx)\Big|_0^{\pi/2} + \frac{1}{4n^2}\underbrace{\sin(2nx)\Big|_0^{\pi/2}}_{0}\right]$$

$$= 8\left[\frac{-\pi/2}{2n}\cos(n\pi) - 0\right] = -\frac{8\pi}{4n}(-1)^n$$

$$= \frac{2\pi}{n}(-1)^{n+1}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2\pi(-1)^{n+1}}{n}\sin(2nx)}$$

5. Find the half-range sine series of $f(x) = \pi$, for $0 < x < \pi$.

$p = \pi$

$$b_n = \frac{2}{\pi} \cdot \int_0^{\pi} f(x) \sin(nx)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \pi \sin(nx)\, dx$$

$$= 2 \left[\frac{-1}{n} \cos(nx) \right]_0^{\pi}$$

$$= \frac{-2}{n} \left[\cos(n\pi) - \cos(0) \right]$$

$$= \frac{-2}{n} \left[(-1)^n - 1 \right] = \frac{2}{n} \left(1 - (-1)^n \right)$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2}{n} \left(1 - (-1)^n \right) \sin(nx)}$$

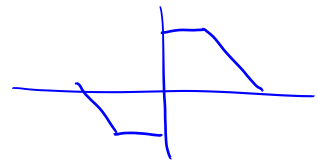
(Extra Credit) Consider the function $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 2 - x, & 1 \leq x < 2 \end{cases}$.

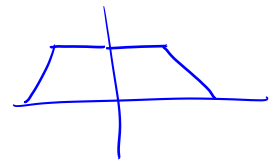
Below are three plots labeled (A), (B) and (C). For each plot, determine if it is the graph of the half-range cosine series of f, the half-range sine series f, or neither of these Fourier series of f.

(A) Cosine

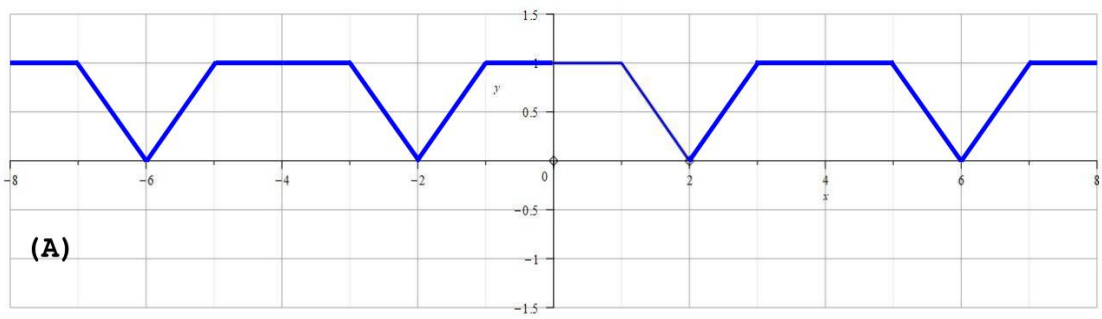
(B) Sine

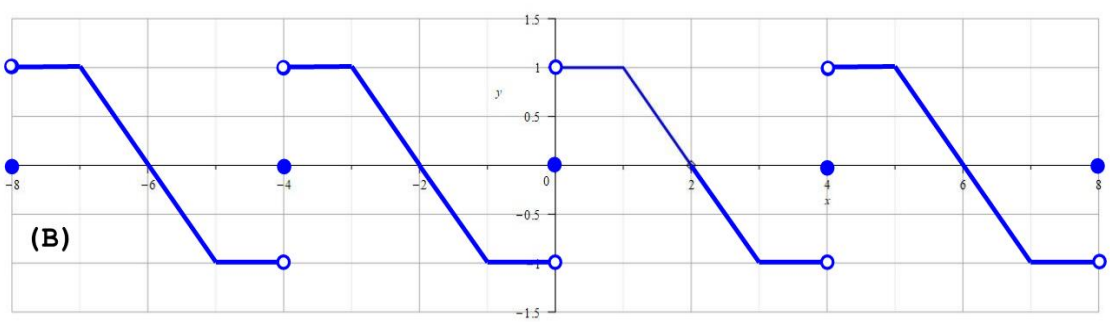
(C) neither

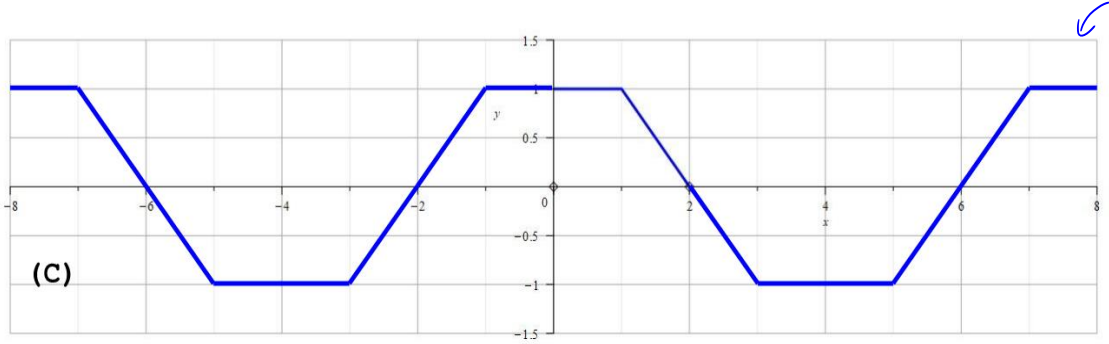




(You can write "cosine," "sine," or "neither.")







← has period 8, should be 4