Exam 4 Math 2306 sec. 54 Spring 2019

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________

Problem	Points
1	
2	
3	
4	
5	

INSTRUCTIONS: There are 5 problems worth 20 points each. You may use one sheet (8.5" $\times$ 11") of your own prepared notes/formulas.
No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.
Show all of your work on the paper provided to receive full credit.

1. Evaluate each Laplace transform.

(a) $\mathscr{L}\left\{t^2\mathscr{U}(t-2)\right\}$ $= e^{-2s}\mathscr{L}\{(t+2)^2\} = e^{-2s}\mathscr{L}\{t^2+4t+4\}$

$= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)$

(b) $\mathscr{L}\left\{e^{-2t}\cos(\pi t)\right\}$ $= \dfrac{s+2}{(s+2)^2+\pi^2}$

2. Evaluate each inverse Laplace transform.

(a) $\mathscr{L}^{-1}\left\{\frac{s}{s^2-8s+20}\right\}$ $= \mathscr{L}^{-1}\left\{\frac{s-4}{(s-4)^2+4}\right\} + 2\mathscr{L}^{-1}\left\{\frac{2}{(s-4)^2+4}\right\}$

$= e^{4t}\cos 2t + 2e^{4t}\sin 2t$

$s^2-8s+20=(s-4)^2+4$

$s=s-4+4$

$4=2\cdot 2$

(b) $\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s}\right\}$ $= \cos(3(t-\pi))\,\mathcal{U}(t-\pi)$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\}=\cos(3t)$

3. Solve the first order initial value problem using the method of Laplace transforms.

$$\frac{dy}{dt} - 2y = 4 + t^2 e^{2t}, \quad y(0) = 1$$

Let $Y(s) = \mathcal{L}\{y(t)\}$

$$\mathcal{L}\{y' - 2y\} = \mathcal{L}\{4 + t^2 e^{2t}\}$$

$$sY(s) - y(0) - 2Y(s) = \frac{4}{s} + \frac{2!}{(s-2)^3}$$

$$(s-2)Y(s) - 1 = \frac{4}{s} + \frac{2!}{(s-2)^3}$$

$$Y(s) = \frac{4}{s(s-2)} + \frac{2}{(s-2)^4} + \frac{1}{s-2}$$

$$\frac{4}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} \Rightarrow 4 = A(s-2) + Bs$$

$s = 0 \quad 4 = -2A \quad A = -2$

$s = 2 \quad 4 = 2B \quad B = 2$

$$Y(s) = \frac{-2}{s} + \frac{2}{s-2} + \frac{2}{(s-2)^4} + \frac{1}{s-2} = \frac{-2}{s} + \frac{3}{s-2} + \frac{2}{(s-2)^4}$$

Note $\mathcal{L}^{-1}\left\{\frac{2}{s^4}\right\} = \frac{2}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{2}{3!}t^3$

so $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$y(t) = -2 + 3e^{2t} + \frac{2}{3!}t^3 e^{2t}$$

$\frac{2}{3!} = \frac{1}{3}$ so we can write

$$y(t) = -2 + 3e^{2t} + \frac{1}{3}t^3 e^{2t}$$

4. Use the method of Laplace transforms to solve the initial value problem.

$$y'' + y = \begin{cases} 0, & 0 \le t < \pi \\ 6, & t \ge \pi \end{cases} \qquad y(0) = 1, \quad y'(0) = 0$$

$$= 6u(t-\pi)$$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{6u(t-\pi)\}$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{6}{s}e^{-\pi s}$$

$$(s^2+1)Y(s) - s = \frac{6}{s}e^{-\pi s}$$

$$Y(s) = \frac{6}{s(s^2+1)}e^{-\pi s} + \frac{s}{s^2+1}$$

$$\frac{6}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \Rightarrow \quad 6 = A(s^2+1) + (Bs+C)s$$

$$= (A+B)s^2 + Cs + A$$

$$A+B=0$$

$$C=0 \quad \Rightarrow B=-A=-6$$

$$A=6$$

$$Y(s) = \frac{6}{s}e^{-\pi s} - \frac{6s}{s^2+1}e^{-\pi s} + \frac{s}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$\boxed{y(t) = 6u(t-\pi) - 6\cos(t-\pi)u(t-\pi) + \cos t}$$

5. Suppose that $f(x) = \begin{cases} 2, & -1 < x < 0 \\ 0, & 0 < x < 1 \end{cases}$

(a) Determine the Fourier series representation of f.

(b) Sketch a graph of the Fourier series (what it converges to) over the interval $(-3, 3)$.

(c) What value does the series converge to when $x = \frac{1}{2}$? When $x = 1$?

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 2\,dx + \int_{0}^{1} 0\,dx = 2x\Big|_{-1}^{0} = 2$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0} 2\cos(n\pi x)\,dx = \frac{2}{n\pi}\sin(n\pi x)\Big|_{-1}^{0}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{-1}^{0} 2\sin(n\pi x)\,dx = \frac{-2}{n\pi}\cos(n\pi x)\Big|_{-1}^{0}$$

$$= \frac{-2}{n\pi}\left[\cos 0 - \cos(n\pi)\right]$$

$$= \frac{-2}{n\pi}\left(1-(-1)^n\right) = \frac{2}{n\pi}\left((-1)^n - 1\right)$$

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left((-1)^n - 1\right)\sin(n\pi x)$$

When $x = \frac{1}{2}$, the series converges to 0

At $x = 1$, it converges to 1