Exam 4 Math 2306 sec. 56

Fall 2017

Name: Solution

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems; the point values are listed with the problems. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) (20 points, 5 each) Determine the Laplace transform or inverse transform as indicated. Use the table provided and any necessary algebra or function identities.

(a) $\mathscr{L}\left\{t^3 e^{-2t}\right\}$ $= \dfrac{3!}{(s+2)^4}$

(b) $\mathscr{L}^{-1}\left\{\dfrac{e^{-3s}}{s-5}\right\}$ $= e^{5(t-3)} \, \mathcal{U}(t-3)$

(c) $\mathscr{L}\left\{t\mathscr{U}(t-\pi)\right\}$ $= e^{-\pi s} \mathcal{L}\{t+\pi\} = e^{-\pi s}\left(\dfrac{1}{s^2} + \dfrac{\pi}{s}\right)$

(d) $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-4)^3}\right\}$ $= \mathcal{L}^{-1}\left\{\frac{1}{2}\dfrac{2!}{(s-4)^3}\right\} = \frac{1}{2} t^2 e^{4t}$

(2) (10 points) A property of Laplace transforms is that

$$\text{if} \quad \mathscr{L}\{f(t)\} = F(s), \quad \text{then} \quad \mathscr{L}\{tf(t)\} = -F'(s).$$

(The prime indicates the derivative of $F(s)$.) Use this to find

$$\mathscr{L}\{t \sin t\} = -\frac{d}{ds} \frac{1}{s^2+1} = -\left(-(s^2+1)^{-2}(2s) \right) = \frac{2s}{(s^2+1)^2}$$

(3) (15 points) Write the given function f in terms of the unit step function $\mathscr{U}$, and use it to find the Laplace transform of f.

$$f(t) = \begin{cases} t, & 0 \leq t < 4 \\ 4, & t \geq 4 \end{cases} = t - t\mathcal{U}(t-4) + 4\mathcal{U}(t-4)$$

$$= t - (t-4)\mathcal{U}(t-4)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{(t-4)\mathcal{U}(t-4)\}$$

$$= \frac{1}{s^2} - \frac{1}{s^2} e^{-4s}$$

(4) (20 points) Solve the first order IVP using the Laplace transform.

$$y' + 3y = 10e^{2t}, \quad y(0) = 6$$

$$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 10\cdot\mathcal{L}\{e^{2t}\} \qquad Y = \mathcal{L}\{y\}$$

$$sY(s) - y(0) + 3Y(s) = \frac{10}{s-2}$$

$$(s+3)Y - 6 = \frac{10}{s-2}$$

$$Y = \frac{10}{(s-2)(s+3)} + \frac{6}{s+3}$$

$$\frac{10}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3} \Rightarrow 10 = A(s+3) + B(s-2)$$

$s = 2 \quad 10 = 5A \quad A = 2$

$s = -3 \quad 10 = -5B \quad B = -2$

$$Y = \frac{2}{s-2} - \frac{2}{s+3} + \frac{6}{s+3}$$

$$= \frac{2}{s-2} + \frac{4}{s+3}$$

$$y = \mathcal{L}^{-1}\{Y\} = 2e^{2t} + 4e^{-3t}$$

$$\boxed{y(t) = 2e^{2t} + 4e^{-3t}}$$

(5) (20 points) Solve the second order IVP using the Laplace transform.

$$y'' - 2y' + y = 12t^2e^t, \quad y(0) = 1, \quad y'(0) = 1$$

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 12\mathcal{L}\{t^2 e^t\}$$

$$s^2 Y - sy(0) - y'(0) - 2\left(sY - y(0)\right) + Y = 12 \frac{2!}{(s-1)^3}$$

$$(s^2 - 2s + 1)Y(s) - s - 1 + 2 = \frac{24}{(s-1)^3}$$

$$(s-1)^2 Y(s) = \frac{24}{(s-1)^3} + s - 1$$

$$Y(s) = \frac{24}{(s-1)^5} + \frac{s-1}{(s-1)^2} = \frac{24}{(s-1)^5} + \frac{1}{s-1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{4!}{(s-1)^5}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$\boxed{y = t^4 e^t + e^t}$$

(6) (15 points) Find the Fourier series of the given function.

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ 2, & 0 \le x < 1 \end{cases}$$

$p = 1$ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$.

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} dx + \int_0^1 2\,dx$$

$$= x \Big|_{-1}^{0} + 2x \Big|_0^1 = (0-(-1)) + 2(1-0) = 3$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0} \cos(n\pi x)\,dx + \int_0^1 2\cos(n\pi x)\,dx$$

$$= \underbrace{\frac{1}{n\pi}\sin(\cancel{n}\pi x)\Big|_{-1}^{0}}_{0} + \underbrace{\frac{2}{n\pi}\sin(\cancel{n}\pi x)\Big|_0^1}_{0} = 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{-1}^{0} \sin(n\pi x)\,dx + \int_0^1 2\sin(n\pi x)\,dx$$

$$= \frac{-1}{n\pi}\cos(n\pi x)\Big|_{-1}^{0} + \frac{-2}{n\pi}\cos(n\pi x)\Big|_0^1$$

$$= \frac{-1}{n\pi}\left[\cos 0 - \cos(-n\pi)\right] - \frac{2}{n\pi}\left[\cos(n\pi) - \cos(0)\right]$$

$$= \frac{-1}{n\pi} + \frac{(-1)^n}{n\pi} - \frac{2(-1)^n}{n\pi} + \frac{2}{n\pi} = \frac{1}{n\pi} - \frac{(-1)^n}{n\pi}$$

$$b_n = \frac{1-(-1)^n}{n\pi}$$

$$\boxed{f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi}\sin(n\pi x)}$$