Exam 4 Math 2306 sec. 57

Fall 2017

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	

INSTRUCTIONS: There are 6 problems; the point values are listed with the problems. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) (20 points, 5 each) Determine the Laplace transform or inverse transform as indicated. Use the table provided and any necessary algebra or function identities.

(a) $\mathscr{L}\left\{t^5 e^{-4t}\right\} = \dfrac{5!}{(s+4)^6}$

(b) $\mathscr{L}^{-1}\left\{\dfrac{e^{-5s}}{s-3}\right\} = e^{3(t-5)} \mathcal{U}(t-5)$

(c) $\mathscr{L}\left\{t\mathscr{U}(t-2\pi)\right\} = e^{-2\pi s} \mathcal{L}\{t+2\pi\}$

$= e^{-2\pi s}\left(\dfrac{1}{s^2} + \dfrac{2\pi}{s}\right)$

(d) $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-3)^4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{3!}\dfrac{3!}{(s-3)^4}\right\} = \dfrac{1}{6}t^3 e^{3t}$

(2) (10 points) A property of Laplace transforms is that

$$\text{if} \quad \mathscr{L}\{f(t)\} = F(s), \quad \text{then} \quad \mathscr{L}\{tf(t)\} = -F'(s).$$

(The prime indicates the derivative of $F(s)$.) Use this to find

$$\mathscr{L}\{t\sin(2t)\} = -\frac{d}{ds}\frac{2}{s^2+4} = -\left(-2(s^2+4)^{-2}(2s)\right) = \frac{4s}{(s^2+4)^2}$$

(3) (15 points) Write the given function f in terms of the unit step function $\mathscr{U}$, and use it to find the Laplace transform of f.

$$f(t) = \begin{cases} t, & 0 \leq t < 6 \\ 6, & t \geq 6 \end{cases} = t - t\mathscr{U}(t-6) + 6\mathscr{U}(t-6)$$

$$= t - (t-6)\mathscr{U}(t-6)$$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{t\} - \mathscr{L}\{(t-6)\mathscr{U}(t-6)\}$$

$$= \frac{1}{s^2} - \frac{1}{s^2}e^{-6s}$$

(4) (20 points) Solve the first order IVP using the Laplace transform.

$$y' + 2y = 15e^{3t}, \quad y(0) = -2$$

$Y = \mathcal{L}\{y\}$

$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 15\,\mathcal{L}\{e^{3t}\}$

$sY(s) - y(0) + 2Y(s) = 15\dfrac{1}{s-3}$

$(s+2)Y + 2 = \dfrac{15}{s-3}$

$Y = \dfrac{15}{(s-3)(s+2)} + \dfrac{-2}{(s+2)}$

$\dfrac{15}{(s-3)(s+2)} = \dfrac{A}{s-3} + \dfrac{B}{s+2} \Rightarrow 15 = A(s+2) + B(s-3)$

$s=3 \quad 5A = 15 \Rightarrow A = 3$

$s=-2 \quad -5B = 15 \Rightarrow B = -3$

$Y = \dfrac{3}{s-3} - \dfrac{3}{s+2} - \dfrac{2}{s+2} = \dfrac{3}{s-3} - \dfrac{5}{s+2}$

$y = \mathcal{L}^{-1}\{Y\} = 3\mathcal{L}^{-1}\left\{\dfrac{1}{s-3}\right\} - 5\mathcal{L}^{-1}\left\{\dfrac{1}{s+2}\right\}$

$$\boxed{y = 3e^{3t} - 5e^{-2t}}$$

(5) (20 points) Solve the second order IVP using the Laplace transform.

$$y'' - 4y' + 4y = 12t^2e^{2t}, \quad y(0) = 1, \quad y'(0) = 2$$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 12\mathcal{L}\{t^2e^{2t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) + 4Y(s) = 12\frac{2!}{(s-2)^3}$$

$$(s^2 - 4s + 4)Y - s - 2 + 4 = \frac{24}{(s-2)^3}$$

$$(s-2)^2 Y(s) = \frac{24}{(s-2)^3} + s - 2$$

$$Y(s) = \frac{24}{(s-2)^5} + \frac{s-2}{(s-2)^2} = \frac{4!}{(s-2)^5} + \frac{1}{s-2}$$

$$y = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{4!}{(s-2)^5}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$\boxed{y = t^4e^{2t} + e^{2t}}$$

(6) (15 points) Find the Fourier series of the given function.

$$f(x) = \begin{cases} 2, & -1 < x < 0 \\ 1, & 0 \le x < 1 \end{cases} \qquad p = 1 \qquad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 2\,dx + \int_0^1 dx$$

$$= 2x \Big|_{-1}^{0} + x \Big|_0^1 = (0 - (-2)) + (1 - 0) = 3$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0} 2\cos(n\pi x)\,dx + \int_0^1 \cos(n\pi x)\,dx$$

$$= \frac{2}{n\pi} \underbrace{\sin(n\pi x)}_{0} \Big|_{-1}^{0} + \frac{1}{n\pi} \underbrace{\sin(n\pi x)}_{0} \Big|_0^1 = 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{-1}^{0} 2\sin(n\pi x)\,dx + \int_0^1 \sin(n\pi x)\,dx$$

$$= \frac{-2}{n\pi}\cos(n\pi x)\Big|_{-1}^{0} + \frac{-1}{n\pi}\cos(n\pi x)\Big|_0^1$$

$$= \frac{-2}{n\pi}\left[\cos 0 - \cos(-n\pi)\right] - \frac{1}{n\pi}\left[\cos(n\pi) - \cos 0\right]$$

$$= \frac{-2}{n\pi} + \frac{2(-1)^n}{n\pi} - \frac{(-1)^n}{n\pi} + \frac{1}{n\pi} = \frac{-1}{n\pi} + \frac{(-1)^n}{n\pi}$$

$$b_n = \frac{(-1)^n - 1}{n\pi}$$

$$\boxed{f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n\pi} \sin(n\pi x)}$$