Exam 4 Math 2306 sec. 58

Spring 2016

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	

INSTRUCTIONS: There are 5 problems worth 20 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

Throughout, take $g = 32$ ft/sec^2 or $g = 9.8$ m/sec^2, which ever is appropriate to the units in context.

(1) A 64 lb object is attached to a spring whose spring constant is 16 lb/ft. It is subject to a damping force that is numerically equal to 12 times the instantaneous velocity.

(a) Find the mass m in slugs.

Weight $W = mg$

$64\text{lb} = m(32\ \text{ft}/\text{sec}^2) \Rightarrow m = \frac{64}{32}$ slugs $= 2$ Slugs

(b) The object is initially displaced 1 ft above equilibrium and given zero initial velocity. Determine its displacement for $t > 0$.

$mx'' + \beta x' + kx = 0$ $\qquad m = 2,\ \beta = 12,\ k = 16$

So $2x'' + 12x' + 16x = 0 \Rightarrow x'' + 6x' + 8x = 0$

$x(0) = 1,\ x'(0) = 0$

Characteristic Eqn: $r^2 + 6r + 8 = 0 \Rightarrow (r+2)(r+4) = 0$

$r = -2$ or $r = -4$

$$\boxed{x(t) = c_1 e^{-2t} + c_2 e^{-4t}}$$

$x(0) = c_1 + c_2 = 1$

$x'(t) = -2c_1 e^{-2t} - 4c_2 e^{-4t}$ $\qquad x'(0) = -2c_1 - 4c_2 = 0$

$$\left.\begin{aligned} 2c_1 + 2c_2 &= 2 \\ -2c_1 - 4c_2 &= 0 \end{aligned}\right\} \Rightarrow -2c_2 = 2 \qquad c_2 = -1$$

$c_1 = 1 - c_2 = 2$

$$\boxed{\text{So} \quad x(t) = 2e^{-2t} - e^{-4t}}$$

(c) Is this motion underdamped, overdamped, or critically damped?

There are two distinct real roots. Hence the system is over damped.

(2) Evaluate each Laplace transform.

(a) $\mathscr{L}\{te^{2t} - \cos(3t)\}$ $= \mathcal{L}\{e^{2t}\, t\} - \mathcal{L}\{\cos(3t)\}$

$$= \frac{1}{(s-2)^2} - \frac{s}{s^2+9}$$

(b) $\mathscr{L}\{e^{-2t}\mathscr{U}(t-4)\}$ $= e^{-4s}\,\mathcal{L}\{e^{-2(t+4)}\} = e^{-4s}\, e^{-8}\,\mathcal{L}\{e^{-2t}\}$

$$= \frac{e^{-8}\, e^{-4s}}{s+2}$$

(c) $\mathscr{L}\{\sin(\pi t) + t^3\}$ $= \mathcal{L}\{\sin(\pi t)\} + \mathcal{L}\{t^3\}$

$$= \frac{\pi}{s^2+\pi^2} + \frac{3!}{s^4} = \frac{\pi}{s^2+\pi^2} + \frac{6}{s^4}$$

(d) $\mathscr{L}\{(e^t+1)^2\}$ $= \mathcal{L}\{e^{2t} + 2e^t + 1\}$

$$= \mathcal{L}\{e^{2t}\} + 2\,\mathcal{L}\{e^t\} + \mathcal{L}\{1\}$$

$$= \frac{1}{s-2} + \frac{2}{s-1} + \frac{1}{s}$$

(3) Find the **general solution** of the nonhomogeneous equation. The complementary solution is provided.

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 6x^3, \quad y_c = c_1x^2 + c_2x^3$$

$y_1 = x^2 \qquad y_2 = x^3 \qquad W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4$

$g(x) = 6x^3$ put $y_p = u_1y_1 + u_2y_2$

variation of parameters

$u_1 = \int -\frac{y_2 g}{W}\,dx = \int \frac{-x^3(6x^3)}{x^4}\,dx = -6\int x^2\,dx = -2x^3$

$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{x^2(6x^3)}{x^4}\,dx = 6\int x\,dx = 3x^2$

$y_p = -2x^3(x^2) + 3x^2(x^3) = x^5$

The general solution is

$$y = c_1 x^2 + c_2 x^3 + x^5.$$

(4) Evaluate each inverse Laplace transform.

(a) $\mathscr{L}^{-1}\left\{\frac{s-1}{s^2+16}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+16}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{4}\ \frac{4}{s^2+16}\right\}$

$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+16}\right\} - \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{4}{s^2+16}\right\}$

$= \cos(4t) - \frac{1}{4}\sin(4t)$

(b) $\mathscr{L}^{-1}\left\{\frac{s}{(s-2)^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{s-2+2}{(s-2)^2+1}\right\}$

$= \mathscr{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1}\right\} + 2\mathscr{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\}$

$= e^{2t}\cos t + 2e^{2t}\sin t$

(c) $\mathscr{L}^{-1}\left\{e^{-2s}\left(\frac{1}{s}+\frac{1}{s-3}\right)\right\}$

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$

$= 1\,\mathcal{U}(t-2) + e^{3(t-2)}\,\mathcal{U}(t-2)$

(5) A LC series circuit has inductance 5 henries and capacitance $\frac{1}{20}$ farads (there is no resistor). A force of $E = 100e^{-t}$ volts is applied. If the initial charge on the capacitor $q(0) = 0$ and the initial current $i(0) = 0$, determine the charge on the capacitor for $t > 0$.

$Lq'' + Rq' + \frac{1}{C} q = E$ Here $L = 5$, $R = 0$, $C = \frac{1}{20}$

$E = 100 e^{-t}$

$5q'' + \frac{1}{1/20} q = 100 e^{-t}$ $q(0) = 0$, $q'(0) = 0$

$5q'' + 20q = 100e^{-t} \Rightarrow q'' + 4q = 20e^{-t}$

Char. Eqn $r^2 + 4 = 0 \Rightarrow r = \pm 2i$ so

$q_c = C_1 \cos(2t) + C_2 \sin(2t)$.

Using Undetermined coefficients $q_p = Ae^{-t}$

$q_p' = -Ae^{-t}$, $q_p'' = Ae^{-t}$

$q_p'' + 4q_p = Ae^{-t} + 4Ae^{-t} = 20e^{-t}$

$5A = 20 \Rightarrow A = 4$

So $q(t) = c_1 \cos(2t) + C_2 \sin(2t) + 4e^{-t}$

$q'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t) - 4e^{-t}$

$q(0) = C_1 + 4 = 0 \Rightarrow C_1 = -4$

$q'(0) = 2C_2 - 4 = 0 \Rightarrow C_2 = 2$

The charge is

$$\boxed{q(t) = -4\cos(2t) + 2\sin(2t) + 4e^{-t}}$$