Exam 4 Math 2306 sec. 59

Spring 2016

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	

INSTRUCTIONS: There are 5 problems worth 20 points each. You may use one sheet (8.5" $\times$ 11") of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

Throughout, take $g = 32$ ft/sec^2 or $g = 9.8$ m/sec^2, which ever is appropriate to the units in context.

(1) A LRC series circuit has inductance 2 henries, resistance 10 ohms, and capacitance $\frac{1}{12}$ farads. A force of $E = 8e^{-t}$ volts is applied. If the initial charge on the capacitor $q(0) = 0$ and the initial current $i(0) = 0$, determine the charge on the capacitor for $t > 0$.

$L q'' + R q' + \frac{1}{C} q = E$ Here, $L = 2$, $R = 10$, $C = \frac{1}{12}$, $E = 8e^{-t}$

$2q'' + 10q' + \frac{1}{1/12} q = 8e^{-t}$ $q(0) = 0$, $q'(0) = 0$

$2q'' + 10q' + 12q = 8e^{-t} \Rightarrow q'' + 5q' + 6q = 4e^{-t}$

Char. Eqn: $r^2 + 5r + 6 = 0 \Rightarrow (r+2)(r+3) = 0$

$r = -2$ or $r = -3$

$q_c = c_1 e^{-2t} + c_2 e^{-3t}$

Using Undetermined Coefficients $q_p = Ae^{-t}$

$q_p' = -Ae^{-t}$, $q_p'' = Ae^{-t}$

$q_p'' + 5q_p' + 6q_p = Ae^{-t} - 5Ae^{-t} + 6Ae^{-t} = 4e^{-t}$

$2A = 4$ $A = 2$

$q(t) = C_1 e^{-2t} + C_2 e^{-3t} + 2e^{-t}$

$q'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t} - 2e^{-t}$

$q(0) = C_1 + C_2 + 2 = 0$ $C_1 + C_2 = -2$

$q'(0) = -2C_1 - 3C_2 - 2 = 0$ $-2C_1 - 3C_2 = 2$

$2C_1 + 2C_2 = -4$

$-C_2 = -2$ $C_2 = 2$

$C_1 = -2 - C_2 = -4$

$$\boxed{\text{So the charge is } q(t) = -4e^{-2t} + 2e^{-3t} + 2e^{-t}}$$

(2) A 16 lb object is attached to a spring whose spring constant is 8 lb/ft. It is subject to a damping force that is numerically equal to 4 times the instantaneous velocity.

(a) Find the mass m in slugs.

Weight $W = mg$

$$m = \frac{W}{g} = \frac{16 \text{ lb}}{32 \frac{\text{ft}}{\text{sec}^2}} = \frac{1}{2} \text{ slug}$$

(b) The object is initially displaced 1/2 ft above equilibrium and given zero initial velocity. Determine its displacement for $t > 0$.

$m x'' + \beta x' + k x = 0 \qquad m = \frac{1}{2}, \quad \beta = 4, \quad k = 8$

$\frac{1}{2} x'' + 4x' + 8x = 0 \implies x'' + 8x' + 16x = 0$

$x(0) = \frac{1}{2}, \quad x'(0) = 0$

Ch. eqn $r^2 + 8r + 16 = 0 \implies (r+4)^2 = 0 \implies r = -4$ repeated

$x(t) = c_1 e^{-4t} + c_2 t e^{-4t} \qquad x'(t) = -4c_1 e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{-4t}$

$x(0) = c_1 = \frac{1}{2} \qquad x'(0) = -4c_1 + c_2 = 0 \qquad c_2 = 4c_1 = 2$

$$\boxed{\text{So } x(t) = \frac{1}{2} e^{-4t} + 2t e^{-4t}.}$$

(c) Is this motion underdamped, overdamped, or critically damped?

There was one repeated real root.

The system is critically damped

(3) Evaluate each inverse Laplace transform.

(a) $\mathscr{L}^{-1}\left\{e^{-2s}\left(\frac{1}{s+4}-\frac{2}{s^3}\right)\right\}$

$\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}$

$\mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2$

$= \mathscr{L}^{-1}\left\{e^{-2s}\frac{1}{s+4}\right\} - \mathscr{L}^{-1}\left\{e^{-2s}\frac{2}{s^3}\right\}$

$= e^{-4(t-2)}\mathcal{U}(t-2) - (t-2)^2\mathcal{U}(t-2)$

(b) $\mathscr{L}^{-1}\left\{\frac{s}{(s-1)^2+4}\right\} = \mathscr{L}^{-1}\left\{\frac{s-1+1}{(s-1)^2+4}\right\}$

$= \mathscr{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{2}\frac{2}{(s-1)^2+4}\right\}$

$= e^t\cos(2t) + \frac{1}{2}e^t\sin(2t)$

(c) $\mathscr{L}^{-1}\left\{\frac{1}{s^2+s}\right\}$

$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \Rightarrow$

$1 = A(s+1) + Bs$

$s=0 \quad 1 = A$

$s=-1 \quad 1 = -B$

$= \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$

$= \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}$

$= 1 - e^{-t}$

(4) Evaluate each Laplace transform.

(a) $\mathscr{L}\{(e^{-t}-1)^2\}$ $= \mathcal{L}\{e^{-2t} - 2e^{-t} + 1\}$

$= \mathcal{L}\{e^{-2t}\} - 2\mathcal{L}\{e^{-t}\} + \mathcal{L}\{1\}$

$= \dfrac{1}{s+2} - \dfrac{2}{s+1} + \dfrac{1}{s}$

(b) $\mathscr{L}\{3t\mathscr{U}(t-4)\}$ $= e^{-4s}\,\mathcal{L}\{3(t+4)\} = e^{-4s}\mathcal{L}\{3t+12\}$

$= 3e^{-4s}\,\mathcal{L}\{t\} + 12e^{-4s}\,\mathcal{L}\{1\}$

$= \dfrac{3e^{-4s}}{s^2} + \dfrac{12e^{-4s}}{s}$

(c) $\mathscr{L}\{t^2e^{2t}+e^t\sin(2t)\}$ $= \mathcal{L}\{t^2e^{2t}\} + \mathcal{L}\{e^t\sin(2t)\}$

$= \dfrac{2}{(s-2)^3} + \dfrac{2}{(s-1)^2+4}$

(d) $\mathscr{L}\{\cos(\pi t)+t^4\}$ $= \mathcal{L}\{\cos(\pi t)\} + \mathcal{L}\{t^4\}$

$= \dfrac{s}{s^2+\pi^2} + \dfrac{4!}{s^5}$

$= \dfrac{s}{s^2+\pi^2} + \dfrac{24}{s^5}$

(5) Find the **general solution** of the nonhomogeneous equation. The complementary solution is provided.

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{1}{x}, \quad y_c = c_1 x + c_2 x \ln(x)$$

$y_1 = x \qquad y_2 = x\ln(x)$

$g(x) = \frac{1}{x}$

$$W = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix} = x\ln x + x - x\ln x = x$$

put $y_p = u_1 y_1 + u_2 y_2$ (Variation of parameters)

$$u_1 = \int -\frac{y_2 g}{W}\,dx = \int \frac{-x\ln x\left(\frac{1}{x}\right)}{x} = -\int \frac{\ln x}{x}\,dx$$

put $v = \ln x$, $dv = \frac{1}{x}dx$

$$= \int -v\,dv = \frac{-v^2}{2} = -\frac{1}{2}(\ln x)^2$$

$$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{x\left(\frac{1}{x}\right)}{x}\,dx = \int \frac{1}{x}\,dx = \ln x$$

$$y_p = -\frac{1}{2}(\ln x)^2 \cdot x + \ln x\,(x\ln x)$$

$$= -\frac{1}{2}x(\ln x)^2 + x(\ln x)^2 = \frac{1}{2}x(\ln x)^2$$

The general solution is

$$\boxed{y = c_1 x + c_2 x\ln x + \frac{1}{2}x(\ln x)^2}$$