Exam 4 Math 2306 sec. 60 Spring 2019

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ____________________

Problem	Points
1	
2	
3	
4	
5	

INSTRUCTIONS: There are 5 problems worth 20 points each. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.
No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.
Show all of your work on the paper provided to receive full credit.

1. Evaluate each Laplace transform.

(a) $\mathscr{L}\left\{t^4e^{6t}\right\} = \dfrac{4!}{(s-6)^5}$

(b) $\mathscr{L}\left\{t^2\mathscr{U}(t-\pi)\right\} = e^{-\pi s}\mathscr{L}\left\{(t+\pi)^2\right\} = e^{-\pi s}\mathscr{L}\left\{t^2+2\pi t+\pi^2\right\}$

$= e^{-\pi s}\left(\dfrac{2!}{s^3}+\dfrac{2\pi}{s^2}+\dfrac{\pi^2}{s}\right)$

2. Evaluate each inverse Laplace transform.

(a) $\mathscr{L}^{-1}\left\{\frac{s}{s^2+4s+8}\right\} = \mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\} - \mathscr{L}^{-1}\left\{\frac{2}{(s+2)^2+4}\right\}$

$= e^{-2t}\cos 2t - e^{-2t}\sin 2t$

$s^2+4s+8 = (s+2)^2+4$

$s = s+2-2$

(b) $\mathscr{L}^{-1}\left\{\frac{e^{-\frac{\pi}{4}s}}{s^2+25}\right\} = \frac{1}{5}\sin\left(5\left(t-\frac{\pi}{4}\right)\right)u\left(t-\frac{\pi}{4}\right)$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{1}{5}\mathscr{L}^{-1}\left\{\frac{5}{s^2+25}\right\} = \frac{1}{5}\sin 5t$

3. Solve the first order initial value problem using the method of Laplace transforms.

$$\frac{dy}{dt} + 3y = 6 + te^{-3t}, \quad y(0) = 1$$

Let $Y(s) = \mathcal{L}\{y(t)\}$

$$\mathcal{L}\{y' + 3y\} = \mathcal{L}\{6 + te^{-3t}\}$$

$$sY(s) - y(0) + 3Y(s) = \frac{6}{s} + \frac{1}{(s+3)^2}$$

$$(s+3)Y(s) - 1 = \frac{6}{s} + \frac{1}{(s+3)^2}$$

$$Y(s) = \frac{6}{s(s+3)} + \frac{1}{s+3} + \frac{1}{(s+3)^3}$$

$$\frac{6}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} \Rightarrow$$

$6 = A(s+3) + Bs$

$s = 0 \quad 6 = 3A \quad A = 2$

$s = -3 \quad 6 = -3B \quad B = -2$

$$Y(s) = \frac{2}{s} - \frac{2}{s+3} + \frac{1}{s+3} + \frac{1}{(s+3)^3}$$

$$= \frac{2}{s} - \frac{1}{s+3} + \frac{1}{(s+3)^3}$$

Note that $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} = \frac{1}{2!}t^2$

As $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$y(t) = 2 - e^{-3t} + \frac{1}{2!}t^2e^{-3t}$$

4. Use the method of Laplace transforms to solve the initial value problem.

$$y'' + y = \begin{cases} 0, & 0 \le t < 3\pi \\ 4, & t \ge 3\pi \end{cases} \qquad y(0) = 1, \quad y'(0) = 0$$

$$= 4u(t-3\pi)$$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{4u(t-3\pi)\}$$

$$s^2 Y(s) - s y(0) - y'(0) + Y(s) = \frac{4}{s} e^{-3\pi s}$$

$$(s^2+1) Y(s) - s = \frac{4}{s} e^{-3\pi s}$$

$$Y(s) = \frac{4}{s(s^2+1)} e^{-3\pi s} + \frac{s}{s^2+1}$$

$$\frac{4}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \Rightarrow 4 = A(s^2+1) + (Bs+C)s$$

$$= (A+B)s^2 + Cs + A$$

$$A+B=0$$
$$C=0 \quad \Rightarrow B = -A = -4$$
$$A=4$$

$$Y(s) = \frac{4}{s} e^{-3\pi s} - \frac{4s}{s^2+1} e^{-3\pi s} + \frac{s}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$\boxed{y(t) = 4u(t-3\pi) - 4\cos(t-3\pi)u(t-3\pi) + \cos t}$$

5. Suppose that $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 < x < \pi \end{cases}$

(a) Determine the Fourier series representation of f.

(b) Sketch a graph of the Fourier series (what it converges to) over the interval $(-3\pi, 3\pi)$.

(c) What value does the series converge to when $x = \frac{\pi}{2}$? When $x = \pi$?

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$ as f is odd

Similarly, $a_n = 0$ as f is odd

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}(-1)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left[\frac{1}{n}\cos(nx)\right]_0^{\pi} = \frac{2}{n\pi}\left[\cos(n\pi) - \cos(0)\right]$$

$$= \frac{2}{n\pi}\left((-1)^n - 1\right)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left((-1)^n - 1\right)\sin(nx)$$

At $\frac{\pi}{2}$, the series converges to -1

At π, it converges to 0