Final Exam Math 2254 sec. 002

Spring 2015

Name: Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________

Problem	Points
1	
2	
3	
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10	
11	

INSTRUCTIONS: There are 11 problems worth 10 points each. You may exclude any one problem, or I will count your best 10 out of 11. No use of notes, books, or calculator is allowed. Illicit use of notes, a book, calculator, or any smart device will result in a grade of zero on this exam and may result in a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) Find a power series representation in the form $\sum_{n=0}^{\infty} c_n x^n$ for the given function and identify its radius of convergence.

$$f(x) = \frac{1}{2+x}$$

$$f(x) = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1/2}{1-\left(\frac{-x}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{-x}{2}\right)^n \qquad \text{if} \quad \left|\frac{-x}{2}\right| < 1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}} \qquad |x| < 2$$

(2) Find the Taylor Polynomial of degree 2 centered at $a = 1$ for the function $f(x) = x^2 \ln(x)$.

$f(x) = x^2 \ln x$ $\qquad$ $f(1) = 0$

$f'(x) = 2x \ln x + x$ $\qquad$ $f'(1) = 1$

$f''(x) = 2\ln x + 2 + 1$ $\qquad$ $f''(1) = 3$

$T_2(x) = x - 1 + \frac{3}{2}(x-1)^2$

(3) Evaluate the limit using any applicable technique. (If you use any special result, e.g. *squeeze theorem, l'Hopital's rule* etc., identify it.)

$$\lim_{x \to 0} \frac{\cos x - e^x}{x} = \text{"}\frac{0}{0}\text{"} \quad \text{l'H rule}$$

$$= \lim_{x \to 0} \frac{\sin x - e^x}{1} = \frac{-1}{1} = -1$$

(4) Evaluate each indefinite integral.

(a) $\displaystyle\int \frac{dx}{x^2+9} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

(b) $\displaystyle\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$

(5) Find the equation of the line tangent to the graph of the parametric curve at the indicated point.

$$x = 2\sin t, \quad y = t^3 + 4t - 1, \quad \text{at} \quad t = 0$$

when $t = 0$, $x = 0$ $y = -1$

$\frac{dx}{dt} = 2\cos t$ $\frac{dy}{dt} = 3t^2 + 4$ $\frac{dy}{dx} = \frac{3t^2+4}{2\cos t}$

Slope $m = \left.\frac{dy}{dx}\right|_{t=0} = \frac{4}{2} = 2$

$y + 1 = 2(x - 0) \Rightarrow y = 2x - 1$

(6) Use logarithmic differentiation to find $\frac{dy}{dx}$ where $y = (\cos x)^x$.

$\ln y = \ln(\cos x)^x = x \ln(\cos x)$

$\frac{1}{y}\frac{dy}{dx} = \ln \cos x + x \frac{-\sin x}{\cos x}$

$\frac{dy}{dx} = y\left[\ln \cos x - x \tan x\right]$

$\frac{dy}{dx} = (\cos x)^x \left[\ln \cos x - x \tan x\right]$

(7) Determine if the improper integral is convergent or divergent. If convergent, find its value.

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x} \right|_1^t$$

$$= \lim_{t\to\infty} \left[-\frac{\ln t}{t} - \frac{1}{t} - \left(\frac{-\ln 1}{1} - \frac{1}{1} \right) \right]$$

$$= 0 - 0 + 1 \quad = 1$$

$$\int \frac{\ln x}{x^2}\,dx$$

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$v = \frac{-1}{x} \qquad dv = x^{-2}\,dx$

$$= -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx$$

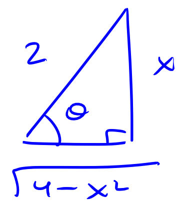
$$= -\frac{\ln x}{x} - \frac{1}{x} + C$$

$$\lim_{t\to\infty} \frac{\ln t}{t} = \text{"}\frac{\infty}{\infty}\text{"} \qquad \text{l'H rule}$$

$$= \lim_{t\to\infty} \frac{\frac{1}{t}}{1} = 0$$

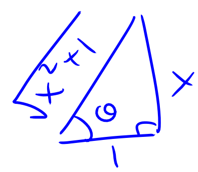
(8) For each integral, draw a representative triangle that can be used to evaluate it when using trigonometric substitution. Identify the trigonometric substitution $x = f(\theta)$. **You are not required to evaluate the integrals.**

(a) $$\int \frac{x^2}{\sqrt{4-x^2}}\,dx$$



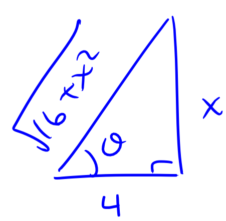
$x = 2\sin\theta$

(b) $$\int \frac{\sqrt{x^2+1}}{2x}\,dx$$



$x = \tan\theta$

(c) $$\int \frac{1}{\sqrt{16+x^2}}\,dx$$



$x = 4\tan\theta$

(9) Evaluate the indefinite integral using any applicable technique.

$$\int \cos^3 x \sin^2 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx$$

$$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int (u^2 - u^4) \, du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

(10) Match each function on the right to its derivative on the left. Put the letter of the function in the correct blank next to its derivative.

$f(x)$		$f'(x)$
(a) $\ln(\cos x)$	b	$3e^{3x}$
(b) e^{3x}	c	$\frac{1}{x\sqrt{x^2-1}}$
(c) $\sec^{-1}(x)$	e	$\frac{1}{\sqrt{x^2-1}}$
(d) $e^{\ln x}$	a	$-\tan x$
(e) $\ln\lvert x + \sqrt{x^2-1}\rvert$	d	1

(11) Use the definition of the Maclaurin series to find the Maclaurin series for the given function and identify its radius of convergence.

$f(x) = 4^x$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f'(x) = 4^x \ln 4$$

$$f''(x) = 4^x (\ln 4)^2$$

$$f'''(x) = 4^x (\ln 4)^3$$

$$\vdots$$

$$f^{(n)}(x) = 4^x (\ln 4)^n$$

$$f^{(n)}(0) = 4^0 (\ln 4)^n = (\ln 4)^n$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(\ln 4)^n}{n!} x^n}$$

$$\lim_{n\to\infty} \left| \frac{(\ln 4)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n (\ln 4)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x \ln 4}{(n+1)} \right| = 0$$

$0 < 1$ for all real x

The radius $R = \infty$.