Final Exam Math 2306 sec. 54

Fall 2015

Name: (1 point) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ________________________________

Problem	Points
1	
2	
3	
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9	
10	

INSTRUCTIONS: There are 10 problems worth 11 points each. You may eliminate any one problem, or I will count your best 9 out of 10. You may use one sheet (8.5" × 11") of your own prepared notes/formulas and the provided table of Laplace transforms. **No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct.** Show all of your work on the paper provided to receive full credit.

(1) Solve the first order initial value problem using any applicable technique.

$$\frac{dy}{dx} + y = 4xe^x \quad y(0) = 2$$

$P(x) = 1 \qquad \mu = e^{\int dx} = e^x$

$$\int \frac{d}{dx}\left(e^x y\right) dx = \int 4x e^x \cdot e^x \, dx$$

$u = 4x \qquad du = 4\,dx$

$$e^x y = 2xe^{2x} - \int 2e^{2x} \, dx$$

$v = \frac{1}{2}e^{2x} \qquad dv = e^{2x}\,dx$

$$= 2xe^{2x} - e^{2x} + C$$

$$y = 2xe^x - e^x + Ce^{-x}$$

$y(0) = -1 + C = 2 \Rightarrow C = 3$

$$\underline{\underline{y = 2xe^x - e^x + 3e^{-x}}}$$

(2) Without attempting to solve any of the following nonhomogeneous equations, determine if it is POSSIBLE to find a particular solution y_p using the method of undetermined coefficients (MUC), using variation of parameters (VP), or both. Assume in all cases that the complementary solution y_c is already known.

(a) $x^2y'' + 2xy' - 3y = e^{2x}$ **MUC** **VP** **Both**

(b) $y'' - 3y = e^{2x}$ **MUC** **VP** **Both**

(c) $2y'' + 3y' - y = x^{-1}\sin x$ **MUC** **VP** **Both**

(3) A 300 gallon tank initially holds fresh water. Brine containing 1 pound of salt per gallon is pumped in at a rate of 3 gallons per minute, and the well mixed solution is pumped out at the same rate. Determine the amount of salt $A(t)$ in pounds at the time t in minutes for all $t > 0$.

$$A(0) = 0$$

$$\frac{dA}{dt} = r_i C_i - r_o C_o = 1 \cdot 3 - 3 \cdot \frac{A}{300}$$

$$\frac{dA}{dt} + \frac{1}{100} A = 3 \qquad \mu = e^{\int \frac{1}{100} dt} = e^{\frac{1}{100} t}$$

$$\int \frac{d}{dt}\left[e^{\frac{1}{100}t} A \right] dt = \int 3 e^{\frac{1}{100}t} dt$$

$$e^{\frac{1}{100}t} A = 300 e^{\frac{1}{100}t} + C$$

$$A = 300 + C e^{\frac{-1}{100}t}$$

$$A(0) = 300 + C = 0 \Rightarrow C = -300$$

$$\boxed{A = 300 - 300 e^{\frac{-1}{100}t}}$$

(4) An LRC series circuit with inductance 1 h, resistance 10 ohms, and capacitance $\frac{1}{125}$ f exhibits free electrical vibrations (i.e. there is no implied force). If the initial charge on the capacitor $q(0) = 0$ C and the initial current $i(0) = 10$ A, find the charge $q(t)$ on the capacitor for $t > 0$.

$$Lq'' + Rq' + \frac{1}{C} q = E$$

$$q'' + 10q' + 125q = 0 \qquad q(0) = 0, \quad q'(0) = 10$$

$$m^2 + 10m + 125 = 0$$

$$m^2 + 10m + 25 + 100 = 0 \Rightarrow (m+5)^2 = -100$$

$$m = -5 \pm 10i$$

$$q(t) = C_1 e^{-5t} \cos(10t) + C_2 e^{-5t} \sin(10t)$$

$$q' = -5C_1 e^{-5t} \cos 10t - 10C_1 e^{-5t} \sin 10t - 5C_2 e^{-5t} \sin 10t + 10C_2 e^{-5t} \cos(10t)$$

$$q(0) = C_1 = 0$$

$$q'(0) = 10C_2 = 10 \Rightarrow C_2 = 1$$

$$\boxed{q(t) = e^{-5t} \sin 10t}$$

(5) Consider the second order nonhomogeneous equation $y'' + 4y = g(x)$. For each possible function g, determine the **form** of the particular solution when using the method of undetermined coefficients. Do not solve for any coefficients A, B, etc.

$$y'' + 4y = 0 \Rightarrow m^2 + 4 = 0 \quad m = \pm 2i$$

$$y_c = C_1 \cos 2x + C_2 \sin 2x$$

(a) $g(x) = x\cos(2x)$

$$y_p = \left[(Ax+B) \cos 2x + (Cx+D) \sin 2x \right] x$$

(b) $g(x) = xe^x$

$$y_p = (Ax+B) e^x$$

(c) $g(x) = 3\sin(2x) - \cos(\pi x)$

$$y_p = (A \sin(2x) + B \cos 2x)x + C \cos \pi x + D \sin \pi x$$

(6) Solve the initial value problem using any applicable technique.

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = -4$$

$$m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1) = 0 \quad m = 2 \text{ or } m = -1$$

$$y = C_1 e^{2x} + C_2 e^{-x}$$

$$y'(x) = 2C_1 e^{2x} - C_2 e^{-x}$$

$$C_1 + C_2 = 1$$

$$2C_1 - C_2 = -4$$

$$3C_1 = -3 \Rightarrow C_1 = -1$$

$$C_2 = 1 - C_1 = 2$$

$$y = -e^{2x} + 2e^{-x}$$

(7) Solve the initial value problem.

$$\begin{aligned} \frac{dx}{dt} &= -3x + 4y \qquad & x(0) = -3 \\ \frac{dy}{dt} &= x \qquad & y(0) = 2 \end{aligned}$$

$(D+3)x - 4y = 0$

$-x + Dy = 0$

$(D^2+3D)x - 4Dy = 0$

$-4x + 4Dy = 0$

$(D^2+3D-4)x = 0$

$m^2+3m-4 = (m+4)(m-1) = 0 \Rightarrow m = -4$ or $m = 1$

$x = C_1 e^{-4x} + C_2 e^{x}$

$Dy = x \Rightarrow y = \frac{-1}{4} C_1 e^{-4x} + C_2 e^{x}$

$\left. \begin{array}{l} x(0) = -3 = C_1 + C_2 \\ y(0) = 2 = \frac{-1}{4} C_1 + C_2 \end{array} \right\} \Rightarrow \quad -5 = \frac{5}{4} C_1 \Rightarrow C_1 = -4$

$C_2 = -3 - C_1 = 1$

$-\frac{1}{4} C_1 = 1$

$x = -4e^{-4x} + e^{x}$

$y = e^{-4x} + e^{x}$

(8) Determine the Laplace transform or inverse transform as indicated.

(a) $\mathscr{L}\left\{\cos(\pi t) + e^t \sin t\right\} = \dfrac{s}{s^2+\pi^2} + \dfrac{1}{(s-1)^2+1}$

(b) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s+2}\right\} = \mathscr{L}^{-1}\left\{\dfrac{s-1}{(s-1)^2+1} - \dfrac{7}{(s-1)^2+1}\right\}$

$= e^t \cos t - 7e^t \sin t$

$s^2 - 2s + 2 = (s-1)^2 + 1$

(c) $\mathscr{L}\left\{\sin(2t)\mathscr{U}(t-1)\right\} = e^{-s}\mathscr{L}\{\sin(2(t+1))\}$

$= e^{-s}\mathscr{L}\{\sin 2t \cos 2 + \sin 2 \cos 2t\}$

$= e^{-s}\dfrac{2\cos 2}{s^2+4} + e^{-s}\dfrac{s \sin 2}{s^2+4}$

(d) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-s-2}\right\} = \mathscr{L}^{-1}\left\{\dfrac{-2}{s+1} + \dfrac{3}{s-2}\right\}$

$= -2e^{-t} + 3e^{2t}$

$\dfrac{s-8}{(s-2)(s+1)} = \dfrac{A}{s-2} + \dfrac{B}{s+1} \Rightarrow s-8 = A(s+1) + B(s-2)$

$s=-1 \quad -9 = -3B \quad \Rightarrow B = 3$

$s = 2 \quad -6 = 3A \quad \Rightarrow A = -2$

(9) Solve the initial value problem using the method of Laplace transforms.

$$y'' + 4y' + 4y = 18te^{-2t}, \quad y(0) = 0, \quad y'(0) = -2$$

$$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{18\,te^{-2t}\}$$

$$s^2Y - sy(0) - y'(0) + 4(sY - y(0)) + 4Y = \frac{18}{(s+2)^2}$$

$$(s^2+4s+4)Y + 2 = \frac{18}{(s+2)^2}$$

$$(s+2)^2 Y = \frac{18}{(s+2)^2} - 2$$

$$Y = \frac{18}{(s+2)^4} - \frac{2}{(s+2)^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \frac{18}{3!}\,t^3 e^{-2t} - \frac{2}{1!}\,t e^{-2t}$$

$$y(t) = 3t^3e^{-2t} - 2te^{-2t}$$

(10) Find the general solution of the second order homogeneous equation for which one solution is given.

$$x^2y'' - 7xy' + 16y = 0, \quad y_1(x) = x^4$$

$$y'' - \frac{7}{x}y' + \frac{16}{x^2}y = 0 \qquad P(x) = \frac{-7}{x}$$

$$y_2 = uy_1 \quad \text{where} \quad u = \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx$$

$$u = \int \frac{e^{\int \frac{7}{x}dx}}{(x^4)^2} = \int \frac{x^7}{x^8}\,dx = \int \frac{1}{x}\,dx = \ln x$$

$$y_2 = x^4 \ln x$$

$$y = c_1 x^4 + c_2 x^4 \ln x$$