Final Exam Math 2306 sec. 59

Spring 2016

Name: (1 point) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
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5	
6	
7	
8	
9	
10	

INSTRUCTIONS: There are 10 problems worth 11 points each. You may eliminate any one problem, or I will count your best 9 out of 10. You may use one sheet (8.5" × 11") of your own prepared notes/formulas.

No use of a calculator, text book, smart device, or other resource is permitted. Illicit use of any additional resource will result in a grade of zero on this exam as well as a formal allegation of academic misconduct. Show all of your work on the paper provided to receive full credit.

(1) Find the general solution of the first order differential equation using any applicable technique.

$$\frac{dy}{dx}+2y = e^{-2x}\sin x$$

$P(x)=2,\quad \int P(x)\,dx = 2x \quad \mu = e^{2x}$

$(e^{2x}y)' = e^{2x}e^{-2x}\sin x = \sin x$

$\int (e^{2x}y)'\,dx = \int \sin x\,dx$

$e^{2x}y = -\cos x + C$

$$\boxed{y = -e^{-2x}\cos x + Ce^{-2x}}$$

(2) Solve the first order IVP using any applicable technique. You may provide an implicit or explicit solution, your choice.

$$\frac{dy}{dx} = (x^2+1)\sec y, \quad y(0) = \frac{\pi}{2}$$

$\frac{dy}{\sec y} = (x^2+1)\,dx$

$\int \cos y\,dy = \int (x^2+1)\,dx$

$\sin y = \frac{x^3}{3} + x + C$

$\sin\left(\frac{\pi}{2}\right) = \frac{0^3}{3} + 0 + C \Rightarrow 1 = C$

$$\boxed{\sin y = \frac{x^3}{3} + x + 1}$$

(3) Solve the initial value problem using any applicable technique.

$y'' - y' - 6y = 0, \quad y(0) = 3, \quad y'(0) = 4$

$m^2 - m - 6 = 0 \qquad (m-3)(m+2) = 0 \qquad m = 3$ or $m = -2$

$y = c_1 e^{3x} + c_2 e^{-2x} \qquad y' = 3c_1 e^{3x} - 2c_2 e^{-2x}$

$c_1 e^0 + c_2 e^0 = 3 \qquad c_1 + c_2 = 3 \qquad 2c_1 + 2c_2 = 6$

$3c_1 e^0 - 2c_2 e^0 = 4 \qquad 3c_1 - 2c_2 = 4 \qquad 3c_1 - 2c_2 = 4$

$5c_1 = 10$

$c_1 = 2$

$c_2 = 3 - c_1 = 1$

$$\boxed{y = 2e^{3x} + e^{-2x}}$$

(4) One solution of the homogeneous equation is given. Find a second, linearly independent solution.

$y'' - \frac{1}{x}y' + 4x^2 y = 0, \quad y_1(x) = \cos(x^2)$

$P(x) = \frac{-1}{x}, \quad -\int P(x)\,dx = \int \frac{1}{x}\,dx = \ln|x|$

$y_2 = u y_1$ where $u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2}\,dx = \int \frac{x}{\cos^2(x^2)}\,dx$

$v = x^2 \quad dv = 2x\,dx$

$y_2 = \frac{1}{2}\tan(x^2)\cos(x^2) \qquad = \int x \sec^2(x^2)\,dx$

$= \frac{1}{2}\sin(x^2) \qquad = \frac{1}{2}\int \sec^2 v\,dv = \frac{1}{2}\tan v = \frac{1}{2}\tan(x^2)$

we can take

$$\boxed{y_2 = \sin(x^2)}$$

(5) Consider the second order nonhomogeneous equation $y'' + 9y = g(x)$. For each possible function g, determine the **form** of the particular solution when using the method of undetermined coefficients. Do not solve for any coefficients A, B, etc.

$$m^2 + 9 = 0 \implies m = \pm 3i$$

$$y_1 = \cos(3x) \qquad y_2 = \sin(3x)$$

$$y_c = C_1 \cos(3x) + C_2 \sin(3x)$$

(a) $g(x) = x\sin(3x)$

$$y_p = \left[(Ax+B)\cos(3x) + (Cx+D)\sin(3x)\right]x$$

$$y_p = (Ax^2+Bx)\cos(3x) + (Cx^2+Dx)\sin(3x)$$

(b) $g(x) = e^{3x} + e^{-3x}$

$$y_p = Ae^{3x} + Be^{-3x}$$

(c) $g(x) = x^2\cos(\pi x)$

$$y_p = (Ax^2+Bx+C)\cos(\pi x) + (Dx^2+Ex+F)\sin(\pi x)$$

(6) An LC series circuit has inductance 1 henry and capacitance 0.1 farads. A constant voltage of 200 volts is applied for 4 seconds and is then turned off. If the initial charge on the capacitor $q(0) = 0$ and the initial current $i(0) = 0$, use Laplace transforms to determine the charge for $t > 0$. Note that q satisfies the IVP

$$\frac{d^2q}{dt^2} + 10q = \begin{cases} 200, & 0 \le t < 4 \\ 0, & t \ge 4 \end{cases} \qquad q(0) = 0, \quad q'(0) = 0$$

$$= 200 - 200\,\mathcal{U}(t-4) \qquad \text{Let } Q(s) = \mathcal{L}\{q(t)\}$$

$$\mathcal{L}\{q'' + 10q\} = \mathcal{L}\{200 - 200\,\mathcal{U}(t-4)\}$$

$$s^2 Q(s) - s\cancel{q(0)}^{\,0} - \cancel{q'(0)}^{\,0} + 10\,Q(s) = \frac{200}{s} - \frac{200}{s}e^{-4s}$$

$$(s^2+10)\,Q(s) = \frac{200}{s} - \frac{200}{s}e^{-4s}$$

$$Q(s) = \frac{200}{s(s^2+10)} - \frac{200}{s(s^2+10)}e^{-4s}$$

Partial fractions: $$\frac{200}{s(s^2+10)} = \frac{A}{s} + \frac{Bs+C}{s^2+10}$$

$$200 = A(s^2+10) + (Bs+C)s$$

$$= (A+B)s^2 + Cs + 10A$$

$$10A = 200 \Rightarrow A = 20$$

$$B = -A \Rightarrow B = -20$$

$$C = 0$$

$$Q(s) = \frac{20}{s} - \frac{20s}{s^2+10} - \frac{20}{s}e^{-4s} + \frac{20s}{s^2+10}e^{-4s}$$

$$q(t) = \mathcal{L}^{-1}\{Q(s)\}$$

$$q(t) = 20 - 20\cos(\sqrt{10}\,t) - 20\,\mathcal{U}(t-4) + 20\cos(\sqrt{10}(t-4))\,\mathcal{U}(t-4)$$

(7) Find the general solution of the second order nonhomogeneous equation. The solutions, $\{y_1, y_2\}$, to the associated homogeneous equation are given.

$$y'' + \frac{1}{x}y' - \frac{9}{x^2}y = 36x, \quad y_1(x) = x^3, \quad y_2(x) = \frac{1}{x^3}$$

assume $x > 0$

$$W = \begin{vmatrix} x^3 & x^{-3} \\ 3x^2 & -3x^{-4} \end{vmatrix} = -3x^3x^{-4} - 3x^2x^{-3} = -6x^{-1}, \quad g(x) = 36x$$

$y_p = u_1y_1 + u_2y_2$

$$u_1 = \int \frac{-y_2 g}{W}\,dx = \int \frac{-x^{-3}(36x)}{-6x^{-1}}\,dx = \int 6x^{-1}\,dx = 6\ln x$$

$$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{x^3(36x)}{-6x^{-1}}\,dx = -6\int x^5\,dx = -x^6$$

$$y_p = (6\ln x)x^3 - x^6(x^{-3}) = 6x^3\ln x - x^3$$

we can lump $-x^3$ in with c_1x^3.

$$\boxed{y = c_1x^3 + c_2x^{-3} + 6x^3\ln x}$$

(8) Use the method of Laplace transforms to solve the IVP. (Note: Solutions not obtained using Laplace transforms will not be considered.)

$$y'' - 5y' + 6y = 18, \quad y(0) = 0, \quad y'(0) = 0$$

Let

$Y(s) = \mathcal{L}\{y(t)\}$

$\mathcal{L}\{y'' - 5y' + 6y\} = \mathcal{L}\{18\}$

$$s^2 Y(s) - s\underset{0}{\underbrace{y(0)}} - \underset{0}{\underbrace{y'(0)}} - 5(sY(s) - \underset{0}{\underbrace{y(0)}}) + 6Y(s) = \frac{18}{s}$$

$$(s^2 - 5s + 6)Y = \frac{18}{s} \Rightarrow Y(s) = \frac{18}{s(s^2 - 5s + 6)}$$

Partial fractions: $\dfrac{18}{s(s-2)(s-3)} = \dfrac{A}{s} + \dfrac{B}{s-2} + \dfrac{C}{s-3}$

$18 = A(s-2)(s-3) + Bs(s-3) + Cs(s-2)$

$s = 0$	$18 = 6A$	$A = 3$
$s = 2$	$18 = -2B$	$B = -9$
$s = 3$	$18 = 3C$	$C = 6$

$$Y(s) = \frac{3}{s} - \frac{9}{s-2} + \frac{6}{s-3}$$

$y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$\boxed{y(t) = 3 - 9e^{2t} + 6e^{3t}}$$

(9) Use the method of Laplace transforms to solve the IVP. (Note: Solutions not obtained using Laplace transforms will not be considered.)

$$y'' - 8y' + 16y = 20t^3e^{4t}, \quad y(0) = -1, \quad y'(0) = -4$$

Let $Y(s) = \mathcal{L}\{y(t)\}$

$$\mathcal{L}\{y'' - 8y' + 16y\} = \mathcal{L}\{20t^3e^{4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 8\left(sY(s) - y(0)\right) + 16Y(s) = 20\frac{3!}{(s-4)^4}$$

$$\left(s^2 - 8s + 16\right)Y(s) + s + 4 - 8 = \frac{120}{(s-4)^4}$$

$$(s^2 - 8s + 16)Y(s) = \frac{120}{(s-4)^4} - s + 4$$

$$(s-4)^2Y(s) = \frac{120}{(s-4)^4} - (s-4)$$

$$Y(s) = \frac{120}{(s-4)^6} - \frac{s-4}{(s-4)^2} = \frac{5!}{(s-4)^6} - \frac{1}{s-4}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$\boxed{y(t) = t^5e^{4t} - e^{4t}}$$

(10) Use the additional information given along with the table of Laplace transforms to find the indicated Laplace transform or inverse transform.

(a) The Laplace transform of $f(t) = \sqrt{t}$ is known. In fact $\mathscr{L}\left\{\sqrt{t}\right\} = \frac{\sqrt{\pi}}{2s^{3/2}}$. Use this to evaluate

$$\mathscr{L}^{-1}\left\{\frac{\sqrt{\pi}}{2(s+1)^{3/2}}\right\}$$

s is replaced w/ s+1

$\mathscr{L}^{-1}\{F(s+1)\} = f(t)e^{-t}$

$= \sqrt{t}\, e^{-t}$

(b) If $\mathscr{L}\{f(t)\} = F(s)$, then $\mathscr{L}\{tf(t)\} = -F'(s)$. Use this to evaluate

$$\mathscr{L}\{t\sin(\pi t)\}$$

$\mathscr{L}\{\sin(\pi t)\} = \frac{\pi}{s^2+\pi^2}$

$\frac{d}{ds}\frac{\pi}{s^2+\pi^2} = -\pi(s^2+\pi^2)^{-2}(2s)$

$= \frac{-2\pi s}{(s^2+\pi^2)^2}$

so $-F'(s) = \frac{2\pi s}{(s^2+\pi^2)^2}$

$= \frac{2\pi s}{(s^2+\pi^2)^2}$