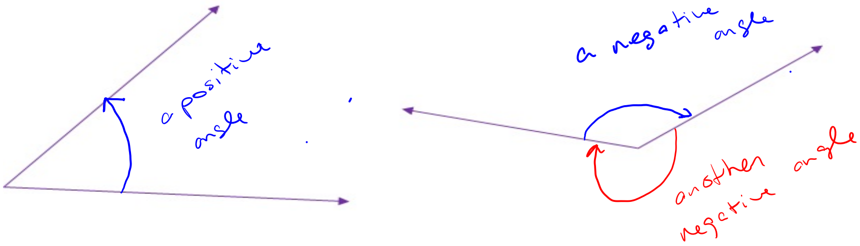
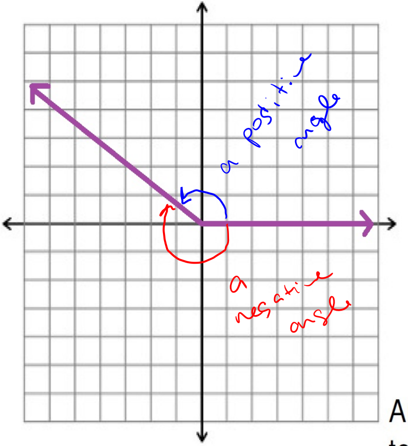
February 10 MATH 1112 sec. 54 Spring 2020

Angles, Rotations, and Angle Measures

We will define an angle by the space, the amount of turn, between two rays that share a common end point (the vertex). We will assign measure and orientation to an angle. A **positive** angle is one that is counter clock-wise. A **negative** angle will be one that is clock-wise.



Angles in Standard Position



- The initial side is always along the $+x$-axis.
- The vertex is at the origin $(0, 0)$.
- The terminal side can go into any quadrant or be along any axis.

Angles in standard position that share a terminal side are called **Co-terminal Angles**.

Degree Measure

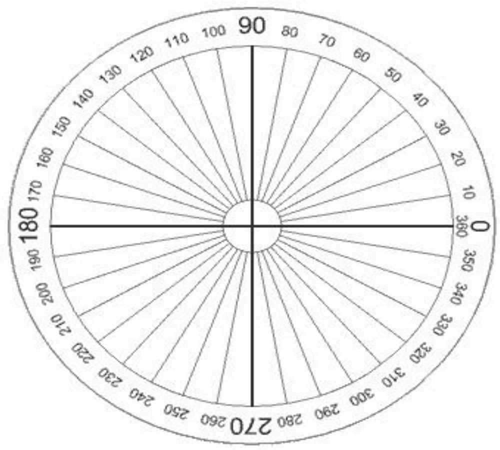


Figure: We can asign a measure to the angle between an initial and terminal side. **Degree** measure is obtained by dividing one full rotation into 360 equal parts.

Coterminal Angles

Since we can measure clockwise (neg.) or counter clockwise (pos.), and can allow for full rotations, angles in standard position may be coterminal but of different measure.

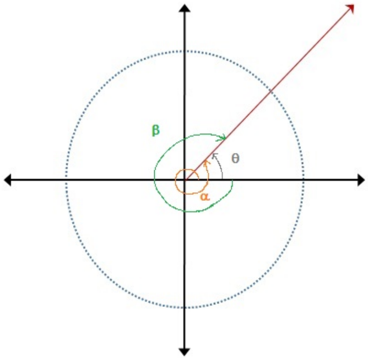


Figure: The three angles θ, α, and β have different measures but are coterminal. **Note: Coterminal angles differ by a multiple of** 360°**.**

Complementary and Supplementary Angles

Definition: Two positive angles whose measures sum to 90° are called **complementary** angles.

Definition: Two positive angles whose measures sum to 180° are called **supplementary** angles.

Example: Find the complementary and the supplementary angles for 38°.

Calling the complement C and the supplement S

$C + 38^\circ = 90^\circ \implies C = 52^\circ$

$S + 38^\circ = 180^\circ \implies S = 142^\circ$

Question

Suppose θ is an angle whose measure is between 0° and 90°. The **complementary** angle to θ is

(a) $\theta + 90^\circ$

(b) $\theta - 90^\circ$

(c) $90^\circ - \theta$

(d) could be any one of the above depending on the actual measure of θ

Radian Measure

In mathematics, we prefer a way to measure angles that is *unitless* [1].

Radians: (Rad) An angle is measured in radians in relation to a unit circle (circle of radius 1).

Central Angle: An angle is called a central angle if its vertex is at the center of a circle.

A central angle measures 1 radian if it subtends an arc of length 1 in a unit circle.

[1] We'll still call them units, but it will become more clear that they aren't units in the traditional sense.

A Radian

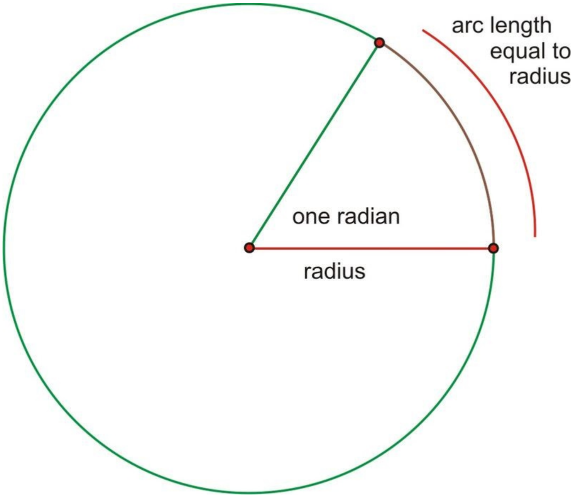


Figure: One Radian: The length of the arc equals the radius of the circle.

Radian Measure

The arc-length of a whole unit circle is 2π. So...

There are 2π radians in one circle (a little more than 6 of them)!

Converting Between Degrees & Radians

Since $360^\circ = 2\pi$ rad, we get the following conversion factors:

$$1^\circ = \frac{\pi}{180} \text{ rad} \qquad \text{and} \qquad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ$$

Remark: If an angle doesn't have the degree symbol $^\circ$ next to it, it is assumed to be in radians!

Converting Between Angle Measures

- To convert from degrees to radians, multiply by

$$\frac{\pi}{180}.$$

- To convert from radians to degrees, multiply by

$$\frac{180}{\pi} \quad \text{and insert the symbol} \quad \circ .$$

Example

Convert each angle measure to the other *units*.

(a) 45°

$45\left(\frac{\pi}{180}\right) = \frac{45\pi}{180} = \frac{\pi}{4}$

(b) $-\frac{\pi}{6}$

$\left(\frac{-\pi}{6} \cdot \frac{180}{\pi}\right)^\circ = \left(-\frac{180}{6}\right)^\circ = -30^\circ$

(b) 30

$\left(30 \, \frac{180}{\pi}\right)^\circ = \left(\frac{5400}{\pi}\right)^\circ$

Question

If $\theta = -210^\circ$, then in radians

(a) $\theta = \frac{7\pi}{6}$

(b) $\theta = -\frac{7\pi}{6}$

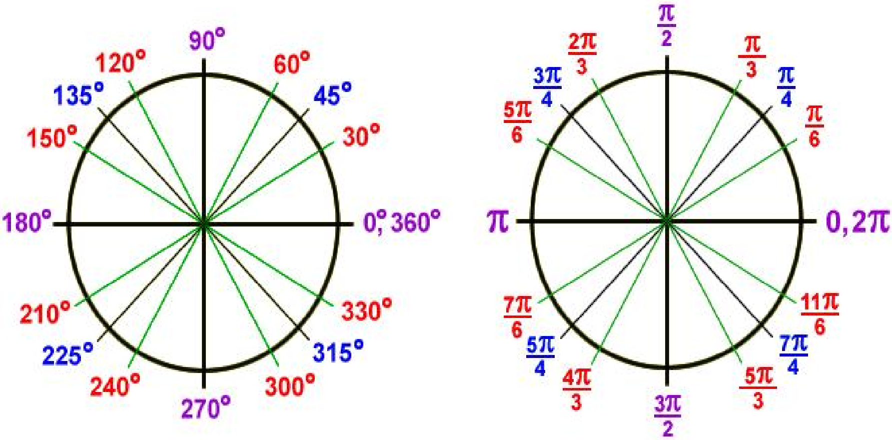
$-210 \left(\frac{\pi}{180}\right) = \frac{-7}{6}\pi$

(c) $\theta = \frac{6\pi}{7}$

(d) $\theta = -\frac{6\pi}{7}$

(e) there's no such thing as a negative angle

Some Common Angles

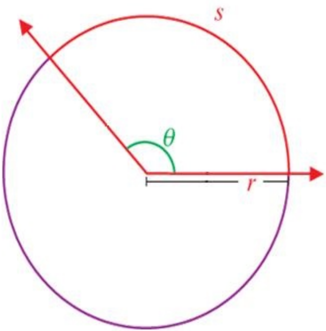


Arclength Formula & Sector Area

Given a circle of radius r, the length s of the arc subtended by the (positive) central angle θ (**in radians**) is given by

$$s = r\theta.$$

The area of the resulting sector is $A_{sector} = \frac{1}{2}r^2\theta.$



Area of sector =

Area of circle × Sector fraction

↓ πr^2 ↓ $\frac{\theta}{2\pi}$

$$A_{sector} = \pi r^2\left(\frac{\theta}{2\pi}\right) = \frac{1}{2}r^2\theta$$

Example

A circle of radius 12 meters has a sector given by a central angle of 135°. Find the associated arc length and the area of the sector.

arclength $s = r\theta$ and Area $A = \frac{1}{2} r^2 \theta$

Here $r = 12$ m. Calling the angle θ, $\theta = 135°$

we need θ in radians.

$$\theta = 135\left(\frac{\pi}{180}\right) = \frac{135\pi}{180} = \frac{3\pi}{4}$$

Arclength $s = (12\text{m})\left(\frac{3\pi}{4}\right) = 9\pi$ m

Area $A = \frac{1}{2}(12\text{m})^2\left(\frac{3\pi}{4}\right) = 72\left(\frac{3\pi}{4}\right)\text{m}^2$

$$= 54\pi \text{ m}^2$$

Question

An industrial clock has a face that is 3 ft in **diameter.** What is the area of the sector between the 12 and the 4 hour markings? (Hint: There are 120° between the 12 and 4 markings.)

(a) $\frac{9\pi}{2}$

(b) $\frac{3\pi}{2}$

(c) $\frac{3\pi}{4}$

(d) 3π

(e) can't be determined without more information

$\theta = 120\left(\frac{\pi}{180}\right) = \frac{2\pi}{3}$

$r = \frac{3}{2}$ ft

$A = \frac{1}{2} r^2 \theta = \frac{1}{2}\left(\frac{3}{2} \text{ ft}\right)^2 \left(\frac{2\pi}{3}\right)$

$= \frac{1}{2} \cdot \frac{9}{2} \cdot \frac{2\pi}{3} \text{ ft}^2$

$= \frac{3\pi}{2} \text{ ft}^2$

Motion on a Circle: Angular & Linear Speed

Definition: (angular speed) If an object moves along the arc of a circle through a central angle θ in the time t, the angular speed is denoted by ω (lower case omega) and is defined by

$$\omega = \frac{\theta}{t} = \frac{\text{angle moved through}}{\text{time}}.$$

Definition: (linear speed) If the circle has radius r, then the distance traveled is the arclength $s = r\theta$. The linear speed is denoted by ν (lower case nu) and is defined by

$$\nu = \frac{s}{t} = \frac{r\theta}{t} = r\omega.$$

Note that this is distance (s) per unit time (t).

Example

A fan blade with a 2 ft radius makes 30 revolutions per minute. Find the linear speed of a point on the outer edge of blade.

We have a radius $r = 2\text{ft}$.

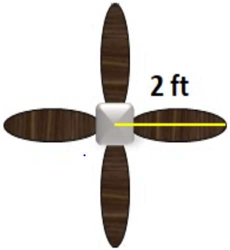
The angular speed is given as

30 revolutions per minute.

Converting this to "radians" per minute

$$\omega = 30 \frac{\text{rev}}{\text{min}} \cdot 2\pi \frac{\text{rad}}{\text{rev}}$$

$$\omega = 60\pi \frac{\text{rad}}{\text{min}}$$



Using $v = r\omega$

$$v = (2\,ft)\left(60\pi\ \frac{rad}{min}\right)$$

$$= 120\pi\ \frac{ft}{min}$$

$$\approx 377\ \frac{ft}{min}$$