Feb. 10 Math 2254H sec 015H Spring 2015

Section 7.8: Improper Integrals

Consider the function $f(t) = \frac{1}{t^2}$. Note that this function is never negative.

What is wrong with the following statement?

$$\int_{-2}^{1} \frac{1}{t^2}\,dt = -\frac{1}{t}\bigg|_{-2}^{1} = -\frac{1}{1} - \left(-\frac{1}{-2}\right) = -\frac{3}{2}$$

FTC: Let f be continuous on [a,b] . . .

Improper Integrals

The integral

$$\int_a^b f(x)\,dx$$

is **improper** if a and/or b is infinite (i.e. $a = -\infty$, $b = \infty$ or both), or if f has an infinite discontinuity at a, b, or somewhere between them (i.e. the graph of f has a vertical asymptote).

The integral **may or may not** have a well defined value. The Fundamental Theorem of Calculus **does not** apply!

Type 1: $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$

Definition: Suppose $\int_a^t f(x)\,dx$ exists for every number $t \geq a$. Then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided the limit exists (as a finite number).

Similarly, if $\int_t^b f(x)\,dx$ exists for every number $t \leq b$. Then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided the limit exists (as a finite number).

Definition Continued...

In either case, if the limit exists, then the integral is said to be **convergent**. Otherwise, it is **divergent**.

If both limits are infinite, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

for any real c provided both integrals on the right are convergent.

Example: Horn of Gabriel

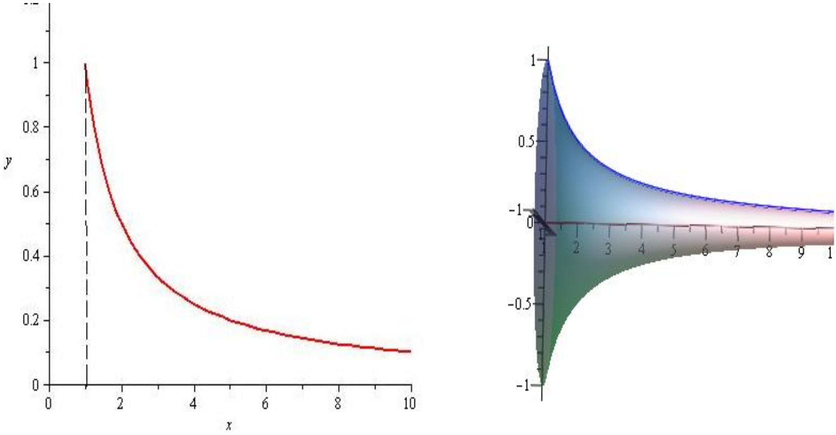
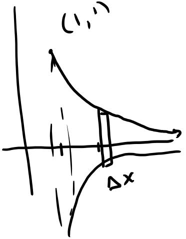


Figure: Consider the region under the curve $f(x) = \frac{1}{x}$ for $1 \leq x < \infty$. Let this be rotated about the x-axis.

Find the Volume of the Horn of Gabriel



Disk

height $= \Delta x$

radius $r = \frac{1}{x}$

$$V_{Disk} = \pi r^2 \Delta x = \pi \left(\frac{1}{x}\right)^2 \Delta x$$

$$= \frac{\pi}{x^2} \Delta x$$

Total volume

$$V = \int_1^\infty \frac{\pi}{x^2}\, dx = \lim_{t \to \infty} \int_1^t \frac{\pi}{x^2}\, dx$$

$$= \lim_{t\to\infty} \left. \frac{-\pi}{x} \right|_1^t$$

$$= \lim_{t\to\infty} \left[\frac{-\pi}{t} - \frac{-\pi}{1} \right] = \pi$$

The horn has finite volume $V = \pi$.

(The integral is convergent.)

Evaluate the Improper Integral if Possible

(a) $\displaystyle\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty} \int_1^t \frac{dx}{x}$

$$= \lim_{t\to\infty} \ln|x| \Big|_1^t$$

$$= \lim_{t\to\infty} \left(\ln|t| - \ln|1|\right) = \infty$$

$\displaystyle\int_1^\infty \frac{dx}{x}$ is divergent.

Evaluate the Improper Integral if Possible

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{1+x^2}$$

$$= \lim_{t\to-\infty} \tan^{-1} x \Big|_{t}^{0}$$

$$= \lim_{t\to -\infty} \left(\tan^{-1} 0 - \tan^{-1} t\right) = 0 - \left(\frac{-\pi}{2}\right) = \frac{\pi}{2}$$

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2} = \lim_{t\to\infty} \tan^{-1} t \Big|_0^t$$

$$= \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} 0\right) = \frac{\pi}{2}$$

Both converge, hence $\int_{-\infty}^\infty \frac{dx}{1+x^2}$ converges

and

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Determine the values of p for which $\int_1^\infty \frac{dx}{x^p}$ converges.

From the previous example, it diverges if $p=1$.

For $p \neq 1$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^p}$$

$$= \lim_{t\to\infty} \left. \frac{x^{1-p}}{1-p} \right|_1^t$$

$$= \lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right) = \begin{cases} \frac{1}{p-1}, & p>1 \\ \infty, & p<1 \end{cases}$$

If $p>1$, $1-p<0$ and

$$\lim_{t\to\infty} t^{1-p} = \lim_{t\to\infty}\frac{1}{t^{p-1}} = 0$$

If $p<1$, $1-p>0$ $\lim_{t\to\infty} t^{1-p} = \infty$

General Result [1]

$$\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1} \quad \text{if } p > 1. \text{ It is divergent if } \; p \le 1.$$

[1] We'll use this again, so keep it in mind!

Type 2: $\int_a^b f(x)\,dx$ with f discontinuous

Definition: If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

provided the limit exists (as a finite number).

Similarly, if f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

provided the limit exists (as a finite number).

Definition Continued...

Again, if the limit exists in either case, the integral is said to be **convergent**. Otherwise it is **divergent**.

If f is continuous on $[a, b]$ except at some point c in (a, b)—-i.e. it is continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

provided each of the integrals on the right side are convergent.

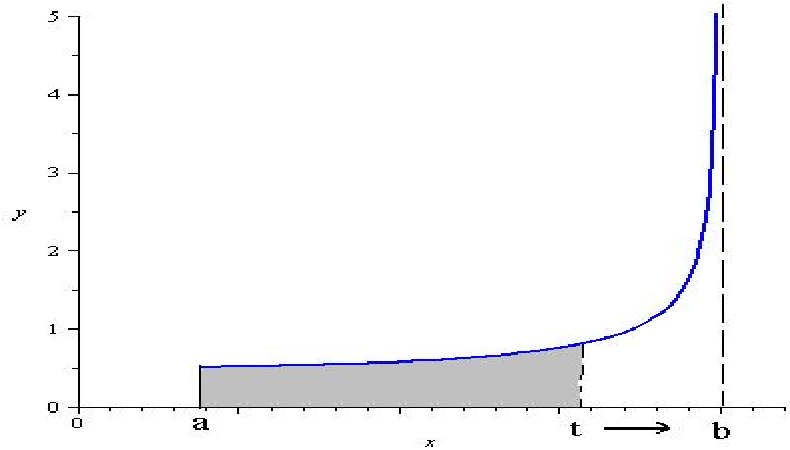


Figure: Area interpretation of improper integral where f has a vertical asymptote at b.

Evaluate the Improper Integral if Possible

(a) $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t \to 0^+} \int_t^1 x^{-1/2}\, dx$

$= \lim_{t \to 0^+} \frac{x^{1/2}}{1/2} \Big|_t^1$

$= \lim_{t \to 0^+} 2\sqrt{x} \Big|_t^1 = \lim_{t \to 0^+} \left(2\sqrt{1} - 2\sqrt{t}\right)$

$= 2$

The integral converges and has value

2.

Evaluate the Improper Integral if Possible

(b) $\displaystyle\int_1^2 \frac{dx}{x-2} = \lim_{t \to 2^-} \int_1^t \frac{dx}{x-2}$

$$= \lim_{t \to 2^-} \ln|x-2| \Big|_1^t$$

$$= \lim_{t \to 2^-} \left(\ln|t-2| - \ln|1-2| \right)$$

$$= -\infty$$

The integral is divergent.

Evaluate the Improper Integral if Possible

(c) $\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \lim_{t\to 3^-} \int_0^t \frac{dx}{\sqrt{9-x^2}}$

$$= \lim_{t\to 3^-} \sin^{-1}\left(\frac{x}{3}\right)\Big|_0^t$$

$$= \lim_{t\to 3^-} \left(\sin^{-1}\left(\frac{t}{3}\right) - \sin^{-1}\left(\frac{0}{3}\right) \right)$$

$$= \sin^{-1}(1) = \frac{\pi}{2}$$

The integral converges w/ value $\frac{\pi}{2}$.

Evaluate the Improper Integral if Possible

(a) $\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x \, dx$

see next slide *

$= \lim_{t\to 0^+} (x\ln x - x)\Big|_t^1$

$= \lim_{t\to 0^+} (1\ln 1 - 1 - [t\ln t - t])$

*

$= -1 - 0 = -1$

$* \quad \int \ln x \, dx$

$u = \ln x \qquad du = \frac{1}{x} dx$

$v = x \qquad dv = dx$

$= x \ln x - \int dx$

$= x \ln x - x + C$

$* \quad \lim_{t \to 0^+} t \ln t = \lim_{t \to 0^+} \frac{\ln t}{1/t} \quad = \text{"} \frac{-\infty}{\infty} \text{"}$

using l'H rule

$= \lim_{t \to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t \to 0^+} -t = 0$