February 11 Math 2306 sec. 54 Spring 2019

Section 5: First Order Equations Models and Applications

A Classic Mixing Problem If we are tracking the amount A of some substance (e.g. salt) disolved in some fluid in which we know the flow rates at which fluid is entering (r_i) and leaving (r_o) a receptacle, the initial volume $V(0)$ of fluid, and the substance concentration of the inflow c_i, then for a **well mixed** solution

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

Here $V(t) = V(0) + (r_i - r_o)t$. This equation is first order linear

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

There is nothing precluding one of the coefficeints such as r_o, r_i or c_i from being a nonconstant function of time.

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

We determined that

$$r_i = 5\,\text{gal/min},\ c_i = 2\,\text{lb/gal},\ r_o = 5\,\text{gal/min},\ ,V(0) = 500\,\text{gal},$$

$A(0) = 0$, and $c_o = \dfrac{A}{500}$. $\qquad V(t) = 500 + (5-5)t = 500$

$$\frac{dA}{dt} + r_o c_o = r_i c_i$$

$$\frac{dA}{dt} + 5\left(\frac{A}{500}\right) = 5 \cdot 2 \qquad A(0) = 0$$

$$\frac{dA}{dt} + \frac{1}{100}A = 10 \qquad P(t) = \frac{1}{100}$$

$$\mu = e^{\int \frac{1}{100}\,dt} = e^{\frac{1}{100}t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) = 10\, e^{\frac{1}{100}t}$$

$$e^{\frac{1}{100}t} A = \int 10\, e^{\frac{1}{100}t}\, dt$$

$$= 10(100)\, e^{\frac{1}{100}t} + k$$

$$A = 1000 + k e^{-\frac{1}{100}t}$$

Apply $A(0) = 0$ $\quad A(0) = 1000 + k e^{0} = 0$

$$1000 + k = 0 \Rightarrow k = -1000$$

$$\boxed{\text{The amount of salt at time } t \text{ is} \quad A(t) = 1000 - 1000 e^{-\frac{1}{100}t}}$$

The concentration in the tank C is

$$C(t) = \frac{A(t)}{V(t)} = \frac{1000 - 1000 e^{-\frac{1}{100}t}}{500} \qquad \frac{\text{lb}}{\text{gal}}$$

At $t=5$ minutes

$$C(5)=\frac{1000-1000e^{-\frac{1}{100}\cdot 5}}{500}\approx 0.0975 \quad \frac{\text{lb}}{\text{gal}}$$

Note that in the long run the concentration

$$\lim_{t\to\infty} C(t)=\lim_{t\to\infty}\frac{1000-1000e^{-\frac{1}{100}t}}{500}$$

$$=\frac{1000}{500}=2 \quad \frac{\text{lb}}{\text{gal}}$$

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

$r_i = 5\ \frac{\text{gal}}{\text{min}} \qquad c_i = 2\ \frac{\text{lb}}{\text{gal}} \qquad r_o = 10\ \frac{\text{gal}}{\text{min}}$

$V(t) = V(0) + (r_i - r_o)t = 500 + (5-10)t = 500 - 5t$

$$\overset{\text{so}}{c_o} = \frac{A}{V} = \frac{A}{500-5t}$$

$$\frac{dA}{dt} + r_o c_o = r_i c_i$$

$$\frac{dA}{dt} + 10\left(\frac{A}{500-5t}\right) = 5\cdot 2$$

$$\frac{dA}{dt} + \frac{2}{100-t} A = 10$$

This is only valid for $0 < t < 100$. Note that $V(100) = 500 - 5(100) = 0$. The volume is zero (tank is empty) at 100 minutes.

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

rate of change $\frac{dP}{dt}$ is proportional to P and proportional to $M-P$.

$$\frac{dP}{dt} = kP(M-P)$$

for some constant of proportionality k.

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**

Solve this equation[2] and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

Let $P(0) = P_0$. The ODE is separable

$$\frac{dP}{dt} = kP(M-P) \Rightarrow \frac{1}{P(M-P)}\frac{dP}{dt} = k$$

$$\int \frac{1}{P(M-P)}\,dP = \int k\,dt$$

[2]The partial fraction decomposition

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

is useful.

$$\int \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)dP = \int k\,dt$$

$$\int \left(\frac{1}{P} + \frac{1}{M-P}\right)dP = \int kM\,dt$$

$$\ln|P| - \ln|M-P| = kMt + C$$

$$\ln\left|\frac{P}{M-P}\right| = kMt + C$$

$$\frac{P}{M-P} = A e^{kMt}$$

where

$A = e^{C}$ or $A = -e^{C}$

Using $P(0)=P_0$ $\quad \dfrac{P_0}{M-P_0}=Ae^0 \Rightarrow \boxed{A=\dfrac{P_0}{M-P_0}}$

Solve for P

$$P=Ae^{kMt}(M-P)$$

$$=AMe^{kMt}-APe^{kMt}$$

$$P+APe^{kMt}=AMe^{kMt}$$

$$(1+Ae^{kMt})P=AMe^{kMt}$$

$$P(t) = \frac{AMe^{kMt}}{1+Ae^{kMt}}$$

use $A = \frac{P_0}{M-P_0}$

$$P(t) = \frac{\frac{P_0}{M-P_0} Me^{kMt}}{1 + \frac{P_0}{M-P_0} e^{kMt}} \cdot \left(\frac{M-P_0}{M-P_0}\right)$$

← clearing fractions

$$\boxed{P(t) = \frac{P_0 M e^{kMt}}{M-P_0+P_0 e^{kMt}}}$$

Solution to the logistic equation.

For $P_0 \neq 0$, we take $t \to \infty$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}}$$

$$= \lim_{t\to\infty} \left(\frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}} \right) \cdot \left(\frac{e^{-kMt}}{e^{-kMt}} \right)$$

$$= \lim_{t\to\infty} \frac{P_0 M}{(M-P_0) e^{-kMt} + P_0}$$

$$= \frac{P_0 M}{(M-P_0)\cdot 0 + P_0} = \frac{P_0 M}{P_0} = M$$