February 11 Math 2306 sec 58 Spring 2016

Section 6: Linear Equations Theory and Terminology

Definition: A set of functions $f_1(x)$, $f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

Definition of Wronskian

Let f_1, $f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

If $y_1, y_2, \ldots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for[1] each x in I.

[1] For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Determine if the functions are linearly dependent or independent:

$$y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty)$$

We'll use the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix}$$

$$= e^x(-2e^{-2x}) - e^x(e^{-2x})$$

$$= -2e^{-x} - e^{-x}$$

$$= -3e^{-x}$$

Note that $W(y_1, y_2)(x) = -3e^{-x} \neq 0$ hence y_1 and y_2 are linearly independent.

* Our $W(x) \neq 0$ for all real x, but it's sufficient that $W(x_0) \neq 0$ for even one x_0.

Fundamental Solution Set

We're still considering this equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

with the assumptions $a_n(x)\neq 0$ and $a_i(x)$ are continuous on I.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,
(ii) there are n of them, and
(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of n^{th} order Linear Homogeneous Equation

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ form a fundamental solution set of the ODE

$$y'' - y = 0 \quad \text{on} \quad (-\infty, \infty),$$

and determine the general solution.

We have 2 functions for a second order equation so property (ii) holds.

Let's verify that they are solutions.

$$y_1 = e^x \qquad y_1'' - y_1 = e^x - e^x = 0$$

$$y_1' = e^x \qquad \text{so } y_1 \text{ is a solution.}$$

$$y_1'' = e^x$$

$y_2 = e^{-x}$

$y_2' = -e^{-x}$

$y_2'' = e^{-x}$

$y_2'' - y_2 = e^{-x} - e^{-x} = 0$

y_2 is also a solution.

Property (i) holds.

Check linear dependence / independence.

Using the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$= e^{x}(-e^{-x}) - e^{x}(e^{-x}) = -2$$

$$W(y_1, y_2)(x) = -2 \neq 0$$

Hence y_1 and y_2 are linearly independent. Property (iii) holds. These form a fundamental solution set.

The general solution is $\boxed{y = c_1 e^{x} + c_2 e^{-x}}$.

Consider $x^2y'' - 4xy' + 6y = 0$ for $x > 0$

Determine which if any of the following sets of functions is a fundamental solution set.

(a) $y_1 = 2x^2, \quad y_2 = x^2$ ← linearly dependent note $1y_1 - 2y_2 = 0$ for all x.

(b) $y_1 = x^2, \quad y_2 = x^{-2}$

(c) $y_1 = x^3, \quad y_2 = x^2$

(d) $y_1 = x^2, \quad y_2 = x^3, \quad y_3 = x^{-2}$ ← too many functions (d) is out.

Does $y=x^2$ solve the Eqn?

$$y_1 = x^2 \qquad x^2 y_1'' - 4x y_1' + 6y_1 =$$

$$y_1' = 2x \qquad x^2(2) - 4x(2x) + 6x^2 =$$

$$y_1'' = 2 \qquad 2x^2 - 8x^2 + 6x^2 = 0$$

yes $y_1 = x^2$ solves the ODE

Does x^{-2} solve it?

$$y_2 = x^{-2} \qquad x^2 y_2'' - 4x y_2' + 6y_2 =$$

$$y_2' = -2x^{-3} \qquad x^2(6x^{-4}) - 4x(-2x^{-3}) + 6x^{-2} =$$

$$y_2'' = 6x^{-4} \qquad 6x^{-2} + 8x^{-2} + 6x^{-2} = 20x^{-2} \neq 0$$

NO $y_2 = x^{-2}$ Does Not solve it

We'll finish this example next time.