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Section 6: Linear Equations Theory and Terminology

Definition: A set of functions $f_1(x)$, $f_2(x), \dots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants c_1, $c_2, \dots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

Definition of Wronskian

Let f_1, $f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

If $y_1, y_2, \ldots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for[1] each x in I.

[1] For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Fundamental Solution Set

We're still considering this equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

with the assumptions $a_n(x) \neq 0$ and $a_i(x)$ are continuous on I.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of n^{th} order Linear Homogeneous Equation

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ form a fundamental solution set of the ODE

$$y'' - y = 0 \quad \text{on} \quad (-\infty, \infty),$$

and determine the general solution.

Note that there are 2 functions and the ODE is 2nd order. Property (ii) holds.

Let's verify that they are solutions.

$y_1 = e^x$

$y_1' = e^x$

$y_1'' = e^x$

$y_1'' - y_1 =$

$e^x - e^x = 0$

y_1 is a solution.

$y_2 = e^{-x}$

$y_2' = -e^{-x}$

$y_2'' = e^{-x}$

$y_2'' - y_2 =$

$e^{-x} - e^{-x} = 0$

y_2 is also a solution.

Property (i) holds.

To check linear independence, we'll use the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$= e^x(-e^{-x}) - e^x(e^{-x}) = -1-1 = -2$$

$$W(y_1, y_2)(x) = -2 \neq 0$$

Hence they are linearly independent. Property (iii) holds. We have a fundamental solution set.

The general solution is $y(x) = c_1 e^x + c_2 e^{-x}$.

Consider $x^2y'' - 4xy' + 6y = 0$ for $x > 0$

Determine which if any of the following sets of functions is a fundamental solution set.

(a) $y_1 = 2x^2, \quad y_2 = x^2$ ← linearly dependent $\quad 1y_1 + (-2)y_2 = 0$

(b) $y_1 = x^2, \quad y_2 = x^{-2}$ ← y_2 doesn't solve it (see below)

(c) $y_1 = x^3, \quad y_2 = x^2$

(d) $y_1 = x^2, \quad y_2 = x^3, \quad y_3 = x^{-2}$ ← too many functions $\quad 2^{nd}$ order

Does $y_1 = x^2$ solve the ODE?

$y_1 = x^2$

$y_1' = 2x$

$y_1'' = 2$

$$x^2 y_1'' - 4x y_1' + 6 y_1 =$$

$$x^2(2) - 4x(2x) + 6x^2 =$$

$$2x^2 - 8x^2 + 6x^2 = 0$$

$y_1 = x^2$ is a solution

Does $y_2 = x^{-2}$ solve the ODE?

$y_2 = x^{-2}$

$y_2' = -2x^{-3}$

$y_2'' = 6x^{-4}$

$$x^2 y_2'' - 4x y_2' + 6 y_2 =$$

$$x^2(6x^{-4}) - 4x(-2x^{-3}) + 6x^{-2} =$$

$$6x^{-2} + 8x^{-2} + 6x^{-2} = 20x^{-2} \neq 0$$

$y_2 = x^{-2}$ is Not a solution

Does $y_1 = x^3$ solve the ODE?

$y_1 = x^3$

$y_1' = 3x^2$

$y_1'' = 6x$

$x^2 y_1'' - 4x y_1' + 6y_1 =$

$x^2(6x) - 4x(3x^2) + 6x^3 =$

$6x^3 - 12x^3 + 6x^3 = 0$

$y_1 = x^3$ does solve the ODE.

Are $y_1 = x^3$, $y_2 = x^2$ linearly independent?

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^3 & x^2 \\ 3x^2 & 2x \end{vmatrix}$$

$$= x^3(2x) - 3x^2(x^2) = 2x^4 - 3x^4 = -x^4$$

$$W(y_1, y_2)(x) = -x^4 \neq 0 \quad (\text{for all } x > 0)$$

Hence y_1, y_2 are linearly independent,

Option (c) is a fundamental solution set.

The general solution is $y = c_1 x^3 + c_2 x^2$.