February 11 Math 2306 sec. 60 Spring 2019

Section 5: First Order Equations Models and Applications

A Classic Mixing Problem If we are tracking the amount A of some substance (e.g. salt) disolved in some fluid in which we know the flow rates at which fluid is entering (r_i) and leaving (r_o) a receptacle, the initial volume $V(0)$ of fluid, and the substance concentration of the inflow c_i, then for a **well mixed** solution

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

Here $V(t) = V(0) + (r_i - r_o)t$. This equation is first order linear

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

There is nothing precluding one of the coefficeints such as r_o, r_i or c_i from being a nonconstant function of time.

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

We determined that

$$r_i = 5\,\text{gal/min},\ c_i = 2\,\text{lb/gal},\ r_o = 5\,\text{gal/min},\ ,V(0) = 500\,\text{gal},$$

$A(0) = 0$, and $c_o = \dfrac{A}{500}$.

$$c_o = \frac{A}{V} = \frac{A}{500+(5-5)t} = \frac{A}{500}$$

$$\frac{dA}{dt} + r_o c_o = r_i c_i$$

Our IVP is $\frac{dA}{dt} + 5\left(\frac{A}{500}\right) = 5 \cdot 2$, $A(0) = 0$

$$\frac{dA}{dt} + \frac{1}{100}A = 10$$

$P(t) = \frac{1}{100}$ so $\mu = e^{\int \frac{1}{100}dt} = e^{\frac{1}{100}t}$

$$\frac{d}{dt}\left(e^{\frac{1}{100}t}A\right) = 10\,e^{\frac{1}{100}t}$$

$$e^{\frac{1}{100}t}A = \int 10\,e^{\frac{1}{100}t}dt = 10(100)\,e^{\frac{1}{100}t} + C$$

$$A = 1000 + C\,e^{-\frac{1}{100}t}$$

Apply $A(0)=0$ $\quad A(0)=1000+Ce^{0}=0$

$1000+C=0 \qquad C=-1000$

$$\boxed{\text{The amount of salt in pounds is} \quad A(t)=1000-1000e^{-\frac{1}{100}t}}$$

The concentration in the tank at t minutes is

$$c(t)=\frac{A(t)}{V(t)}=\frac{A(t)}{500}\ \frac{\text{lb}}{\text{gal}}$$

At $t = 5$ minutes

$$C(5) = \frac{A(5)}{500} = \frac{1000 - 1000e^{-\frac{1}{100}(5)}}{500} \approx 0.0975 \ \frac{\text{lb}}{\text{gal}}$$

Note that as $t \to \infty$

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} \left(1000 - 1000e^{-\frac{1}{100}t}\right)$$

$$= 1000 \quad \text{lb}$$

The concentration tends to $\dfrac{1000 \text{ lb}}{500 \text{ gal}} = 2 \ \dfrac{\text{lb}}{\text{gal}}$

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

As before $r_i = 5\,\frac{\text{gal}}{\text{min}}$, $c_i = 2\,\frac{\text{lb}}{\text{gal}}$, $A(0) = 0$

$r_o = 10\,\frac{\text{gal}}{\text{min}}$ so $V(t) = V(0) + (r_i - r_o)t$

$$= 500 + (5-10)\,t$$

$$= 500 - 5t$$

$$c_o = \frac{A}{V} = \frac{A}{500-5t}$$

$$\frac{dA}{dt} + r_o c_o = r_i c_i$$

$$\frac{dA}{dt} + 10\left(\frac{A}{500-5t}\right) = 5 \cdot 2$$

$$\frac{dA}{dt} + \frac{2}{100-t} A = 10$$

The tank is empty after 100 minutes since $V(100) = 500 - 5(100) = 0$. The equation is valid for $0 < t < 100$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

rate of change $\frac{dP}{dt}$ is proportional to P and the difference $M-P$. The ODE is

$$\frac{dP}{dt} = k P(M-P)$$ for some constant of proportionality k

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation
$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$
is called a **logistic growth equation.**

Solve this equation[2] and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

Let $P(0) = P_0 \neq 0$. The ODE is separable.

$$\frac{dP}{dt} = kP(M-P) \implies \frac{1}{P(M-P)}\frac{dP}{dt} = k$$

$$\int \frac{1}{P(M-P)}\,dP = \int k\,dt$$

[2] The partial fraction decomposition
$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$
is useful.

$$\int \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int k \, dt$$

$$\int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int kM \, dt$$

$$\ln|P| - \ln|M-P| = kMt + C$$

$$\ln\left|\frac{P}{M-P}\right| = kMt + C$$

$$\frac{P}{M-P} = A e^{kMt}$$ where $A = e^{c}$ or $A = -e^{c}$

Using $P(0) = P_0$ $$\frac{P_0}{M-P_0} = Ae^{0}$$

So $\boxed{A = \frac{P_0}{M - P_0}}$ Solve for P

$$\frac{P}{M-P} = Ae^{kMt} \Rightarrow P = Ae^{kMt}(M-P)$$

$$P = AMe^{kMt} - PAe^{kMt}$$

$$P + PAe^{kMt} = AMe^{kMt}$$

$$(1 + Ae^{kMt})P = AMe^{kMt}$$

$$P = \frac{AMe^{kMt}}{1 + Ae^{kMt}}$$

Using $A = \dfrac{P_0}{M - P_0}$

Clear the fractions

$$P = \frac{\frac{P_0}{M-P_0} M e^{kMt}}{1 + \frac{P_0}{M-P_0} e^{kMt}} \cdot \left(\frac{M-P_0}{M-P_0}\right)$$

$$\boxed{P(t) = \frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}}}$$

general solution to the logistic equation

We'll take the limit as $t \to \infty$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{P_0 M e^{kMt}}{M - P_0 + P_0 e^{kMt}} = \text{"}\frac{\infty}{\infty}\text{"}$$

use l'Hopital's rule

$$= \lim_{t\to\infty} \frac{\cancel{P_0} M \cancel{(kM)} \cancel{e^{kMt}}}{\cancel{P_0} \cancel{(kM)} \cancel{e^{kMt}}}$$

$$= \lim_{t\to\infty} M \qquad = M$$