February 4 Math 2335 sec 51 Spring 2016

Section 3.3: Secant Method

Newton's method begins with a straight line approximation to the function $f(x)$—namely, the tangent line.

Question: Can we use a different straight line?

The short answer is "yes" we can. The tangent line touches a curve (locally) at only one point. We recall...

Definition: If the graph of f contains the distinct points $(x_0, f(x_0))$ and $(x_1, f(x_1))$, then the line

$$y = f(x_1) + (x - x_1)\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

is a **secant line** to the graph of f through these points.

Secant Method

We begin with two initial estimates x_0 and x_1 of the true root α.

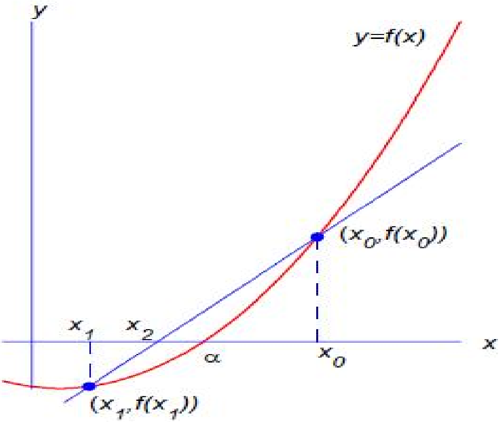

Figure: Choose x_2 as the x-intercept of the secant line approximation.

Secant Method

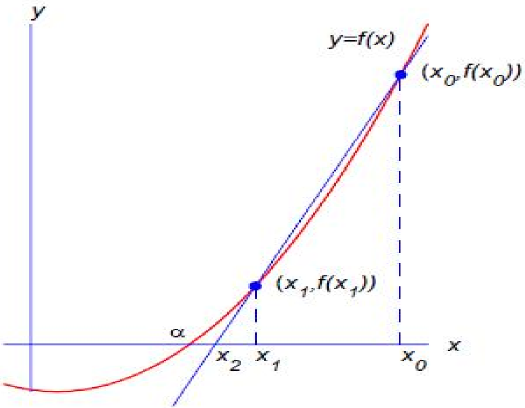

Figure: The starting values x_0 and x_1 can each be on either side of the exact root.

Secant Method

Find the formula for x_2 from the secant line.

$$y = f(x_1) + (x - x_1)\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

When $x = x_2$, $y = 0$

$$0 = f(x_1) + (x_2 - x_1)\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$(x_2 - x_1)\frac{f(x_1) - f(x_0)}{x_1 - x_0} = -f(x_1)$$

$$x_2 - x_1 = -f(x_1) \cdot \frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

assuming

$f(x_1) \neq f(x_0)$

$$\boxed{x_2 = x_1 - f(x_1) \cdot \frac{x_1 - x_0}{f(x_1) - f(x_0)}}$$

Secant Method Compared to Newton's Method

Newton's Method: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

Remember that by the definition of the derivative

$$f'(x_1) = \lim_{x_0 \to x_1} \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

So

$$f'(x_1) \approx \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \text{if} \quad |x_1 - x_0| \approx 0.$$

Secant Method: $x_2 = x_1 - f(x_1) \cdot \dfrac{x_1 - x_0}{f(x_1) - f(x_0)}$

Secant Method Iteration Formula

We build a sequence with the general formula...

Secant Method Iteration Formula

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad n = 1, 2, 3, \ldots$$

The sequence begins with two starting *guesses* x_0 and x_1 expected to be near the desired root.

Exit Strategy: Set an allowable tolerance ϵ, and then stop the iterations if

- $|x_{n+1} - x_n| < \epsilon$, or
- $n \geq N$ where N is the maximum allowed iterations.

If the latter condition is used, it likely indicates that the method has failed.

Example

(a) We wish to compute $\sqrt[3]{4}$. Identify a convenient function $f(x)$ whose zero $\alpha = \sqrt[3]{4}$, and find the iteration formula for the secant method.

$f(x) = x^3 - 4$ has $\alpha = \sqrt[3]{4}$ as its true root.

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

$f(x_n) = x_n^3 - 4$ and $f(x_{n-1}) = x_{n-1}^3 - 4$

$$f(x_n) - f(x_{n-1}) = x_n^3 - 4 - (x_{n-1}^3 - 4) = x_n^3 - x_{n-1}^3$$

So

$$x_{n+1} = x_n - (x_n^3 - 4)\,\frac{x_n - x_{n-1}}{x_n^3 - x_{n-1}^3}$$

Example Continued...

(b) Set $x_0 = 1$ and $x_1 = 2$ and use the iteration scheme to compute x_2.

$$x_2 = x_1 - (x_1^3 - 4)\,\frac{x_1 - x_0}{x_1^3 - x_0^3}$$

$$= 2 - (2^3 - 4)\cdot\frac{2-1}{2^3 - 1^3} = 2 - 4\cdot\frac{1}{7} = \frac{10}{7}$$

Example Continued...

(c) Use $x_1 = 2$ and x_2 found at the last step to compute x_3.

$$x_3 = x_2 - (x_2^3 - 4)\,\frac{x_2 - x_1}{x_2^3 - x_1^3}$$

$$= \frac{10}{7} - \left(\left(\frac{10}{7}\right)^3 - 4\right) \cdot \frac{10/7 - 2}{\left(\frac{10}{7}\right)^3 - 2^3} = \frac{169}{109}$$

$$\doteq 1.550459$$

Example Continued...

The root was found to within an error tolerance of $\epsilon = 10^{-8}$ in 7 steps using Matlab ®.

n	x_n	$f(x_n)$
0	1.000000000000000	−3.000000000000000
1	2.000000000000000	4.000000000000000
2	1.428571428571429	−1.084548104956268
3	1.550458715596330	−0.272817828789934
4	1.591424324468624	0.030491183856831
5	1.587306115447955	−0.000717632200947
6	1.587400811747808	−0.000001815952090
7	1.587401051982567	0.000000000108610
8	1.587401051968199	−0.000000000000001

Error Analysis: Secant Method

Assume that $f'(\alpha) \neq 0$. It can be shown that the errors at the $(n+1)^{st}$ and n^{th} steps are related by

$$|\alpha - x_{n+1}| \approx c|\alpha - x_n|^r$$

$$\text{where} \quad r = \frac{1+\sqrt{5}}{2}, \quad \text{and} \quad c = \left|\frac{f''(\alpha)}{2f'(\alpha)}\right|^{r-1}.$$

Comparison: Newton's & Secant Methods

- Newton's is a **one step** method because x_{n+1} depends only on x_n. Secant is a **two step** method because x_{n+1} depends on x_n and x_{n-1}.

- For Newton's method, we must have a formula for $f'(x)$. For the secant method, this is not needed.

- For both methods, initial guesses may have to satisfy

$$|\alpha - x_i| < \frac{2|f'(\alpha)|}{|f''(\alpha)|}, \quad \text{where} \quad i = 0, \text{ or } i = 0, 1$$

- For Newton's, $|\mathrm{Err}(x_{n+1})| \sim |\mathrm{Err}(x_n)|^2$. whereas for the Secant method $|\mathrm{Err}(x_{n+1})| \sim |\mathrm{Err}(x_n)|^{1.62}$. So Newton's method may require fewer iterations.

Section 3.5: Ill-Behavior in Root Finding

We assumed that $f'(\alpha) \neq 0$ in our error analysis of Newton's and Secant methods. The result was that

$$\text{for Newton's:} \quad |\text{Err}(x_{n+1})| \sim |\text{Err}(x_n)|^2, \quad \text{and}$$

$$\text{for Secant:} \quad |\text{Err}(x_{n+1})| \sim |\text{Err}(x_n)|^{1.62}.$$

If we started with an initial error of 0.1, we'd expect to see error along the lines of

n	Newton's	Secant
0	0.1000000000000000	0.100000000000000
1	0.0100000000000000	0.024097168320749
2	0.0001000000000000	0.002409716832075
3	0.0000000100000000	0.000058067352108
4	0.0000000000000001	0.000000139925876

Ill Behaved Root Finding Example

The function $f(x) = x^3 - 1.4x^2 - 21.75x + 51.2$ has one real root. (It's exact value is $\alpha = 3.2$.) Newton's method was used with an initial guess of 3.1 and a tolerance of $\epsilon = 10^{-3}$ to produce the following table:

n	x_n	$\lvert x_{n+1} - x_n \rvert$	$\lvert \alpha - x_n \rvert$
0	3.100000000000000	0.050310559006211	0.1000
1	3.150310559006211	0.024920686619802	0.0497
2	3.175231245626013	0.012403166317891	0.0248
3	3.187634411943904	0.006187466419181	0.0124
4	3.193821878363085	0.003090225829813	0.0062
5	3.196912104192898	0.001544238772135	0.0015
6	3.198456342965033	0.000771901186694	0.0008
7	3.199228244151727		

Ill Behaved Root Finding Example

The results aren't very good! Let's notice that

$$f(x) = x^3 - 1.4x^2 - 21.75x + 51.2 = (x - 3.2)^2(x + 5).$$

Find $f'(3.2)$.

$$f'(x) = 2(x-3.2)(x+5) + (x-3.2)^2$$

$$f'(3.2) = 2(3.2-3.2)(3.2+5) + (3.2-3.2)^2$$

$$= 0$$

Here $f'(\alpha) = 0$

Ill Behaved Root Finding

Definition: (Multiple Roots) The number α is a root (or zero) of multiplicity m of the function $f(x)$ if

$$f(x) = (x-\alpha)^m h(x), \quad \text{where} \quad h(\alpha) \neq 0.$$

If $m = 1$, we call α a **simple root.**

If f is sufficiently differentiable, and α is a root of multiplicity m of f, then

$$f(\alpha) = f'(\alpha) = \cdots f^{(m-1)}(\alpha) = 0 \quad \text{and} \quad f^{(m)}(\alpha) \neq 0.$$

Examples of Multiple Roots

(a) $f(x) = (x - 3.2)^2(x + 5)$

f has 2 roots

$\alpha_1 = 3.2$ of multiplicity 2

$\alpha_2 = -5$ is a simple root

(b) $g(x) = (x + 1)(x - 1)^3(x - 2)$

g has 3 roots

$\alpha_1 = -1$ and $\alpha_3 = 2$ are simple

$\alpha_2 = 1$ has multiplicity 3

Examples of Multiple Roots

(c) $\alpha = 0$ is a root of multiplicity m of $h(x) = \cos x - 1 + \frac{x^2}{2}$. Find m.

$$h(0) = \cos 0 - 1 + \frac{0^2}{2} = 1 - 1 = 0$$

$$h'(x) = -\sin x + x \quad , \quad h'(0) = -\sin 0 + 0 = 0$$

$$h''(x) = -\cos x + 1 \quad , \quad h''(0) = -\cos 0 + 1 = -1 + 1 = 0$$

$$h'''(x) = \sin x \quad , \quad h'''(0) = 0$$

$$h^{(4)}(x) = \cos x \quad , \quad h^{(4)}(0) = \cos 0 = 1 \neq 0$$

$h^{(i)}(0) = 0$ for $i = 0, 1, 2, 3$

and $h^{(4)}(0) \neq 0$

Hence 0 is a root of multiplicity

4.

Newton's Method with Multiple Roots

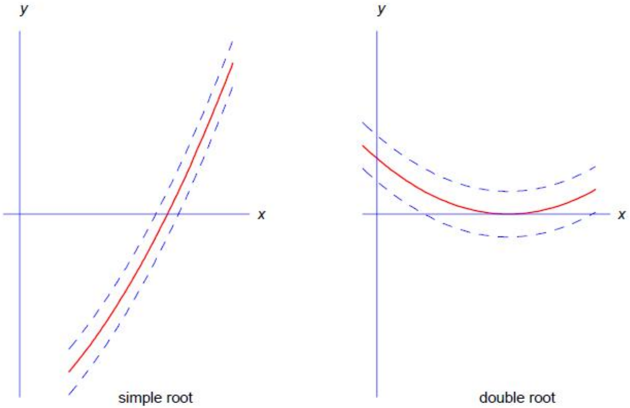

Figure: Noise in function evaluation and a horizontal tangent increases error in root finding for multiple roots.

Newton's Method with Multiple Roots

For a simple root (not a multiple root):

$$\frac{|\alpha - x_{n+1}|}{|\alpha - x_n|} \sim |\alpha - x_n|$$

If α is a root of multiplicity $m \geq 2$ of $f(x)$, then

$$\frac{|\alpha - x_{n+1}|}{|\alpha - x_n|} \sim \lambda, \quad \text{where} \quad \lambda = \frac{m-1}{m}$$

Example

Recall the errors when Newton's method was used with
$f(x) = (x - 3.2)^2(x + 5)$.

n	$\lvert\alpha - x_n\rvert$	$\frac{\lvert\alpha - x_{n+1}\rvert}{\lvert\alpha - x_n\rvert}$
0	0.1000	0.4969
1	0.0497	0.4985
2	0.0248	0.4992
3	0.0124	0.4996
4	0.0062	0.4998
5	0.0031	0.4999
6	0.0015	0.5000
7	0.0008	

Note $m = 2$

$$\lambda = \frac{m-1}{m} = \frac{2-1}{2} = \frac{1}{2}$$

Example

Newton's method was used to try to find a root α of a function $f(x)$. The first several iterates were recorded in the following table. Use these results to make a conjecture as to the multiplicity m of the root α

n	x_n	$\lvert x_{n+1} - x_n \rvert$
0	0.75	0.00271
1	0.752710	0.00208
2	0.754795	0.00157
3	0.756368	0.00118
4	0.757552	0.000889
5	0.758441	

We can approximate

$$\frac{|\alpha - x_{n+1}|}{|\alpha - x_n|} \quad \text{with} \quad \frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}$$

$$\frac{|x_2 - x_1|}{|x_1 - x_0|} \doteq 0.768$$

$$\frac{|x_3 - x_2|}{|x_2 - x_1|} \doteq 0.755 \qquad \frac{|x_4 - x_3|}{|x_3 - x_2|} \doteq 0.752$$

$$\frac{|x_5 - x_4|}{|x_4 - x_3|} \doteq 0.753$$

The ratio is about 0.75

$$\frac{m-1}{m} \approx 0.75 = \frac{3}{4}$$

We expect that α is a root of multiplicity 4.