Section 7.8: Improper Integrals

Show that the integral is divergent

$$\int_{-2}^{1} \frac{1}{x^2}\,dx = \int_{-2}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

Consider the right one:

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \left.\frac{-1}{x}\right|_t^1$$

$$= \lim_{t\to 0^+} \left(\frac{-1}{1} - \frac{-1}{t}\right) = \infty$$

Since $\int_0^1 \frac{1}{x^2}\,dx$ diverges, the

integral $\int_{-2}^1 \frac{1}{x^2}\,dx$ is divergent.

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$$

Independence of Limts

Show that $\displaystyle\int_{-2}^{1} \frac{dx}{x^2} \neq \lim_{t\to 0}\left(\int_{-2}^{t} \frac{dx}{x^2} + \int_{t}^{1} \frac{dx}{x^2}\right).$

$$\lim_{t\to 0}\left(\int_{-2}^{t} \frac{1}{x^2}\,dx + \int_{t}^{1} \frac{1}{x^2}\,dx\right)$$

$$= \lim_{t\to 0}\left[\left.\frac{-1}{x}\right|_{-2}^{t} + \left.\frac{-1}{x}\right|_{t}^{1}\right]$$

$$= \lim_{t\to 0}\left[\frac{-1}{t} - \left(\frac{-1}{-2}\right) + \left(\frac{-1}{1} - \frac{-1}{t}\right)\right] = \lim_{t\to 0}\left(\frac{-1}{2} - 1\right) = \frac{-3}{2}$$

Comparisons

Evaluate the improper integral $\int_1^\infty e^{-x}\,dx$

$$= \lim_{t\to\infty} \int_1^t e^{-x}\,dx = \lim_{t\to\infty} -e^{-x}\Big|_1^t$$

$$= \lim_{t\to\infty} \left(-e^{-t} - (-e^{-1})\right) = \frac{1}{e}$$

Convergent

Is it possible to evaluate $\displaystyle\int_1^\infty e^{-x^2}\,dx$?

There is no elementary antiderivative for e^{-x^2}.

Comparisons

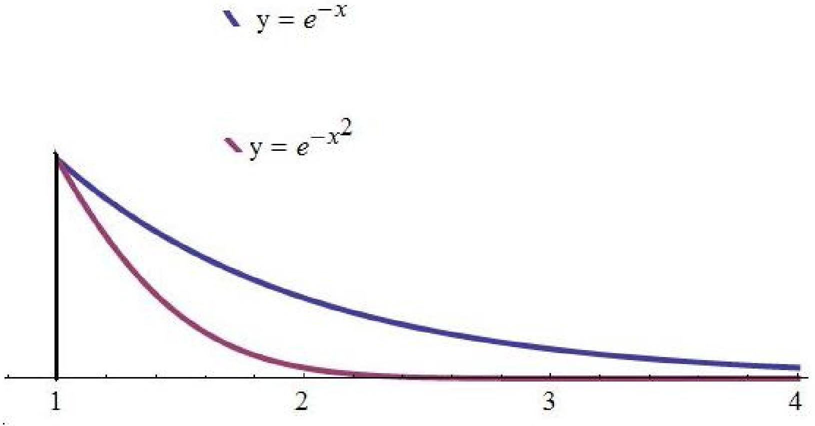

Figure: If the area under the blue curve is finite, can the area under the red curve be infinite?

Comparison Theorem

Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then

(a) if $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.

(b) if $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Example

Determine if the integral

$$\int_1^{\infty} e^{-x^2}\,dx$$

is convergent or divergent.

This is convergent by comparison to $\int_1^{\infty} e^{-x}\,dx$ since

$$0 \leq e^{-x^2} \leq e^{-x} \text{ for all } x \geq 1.$$

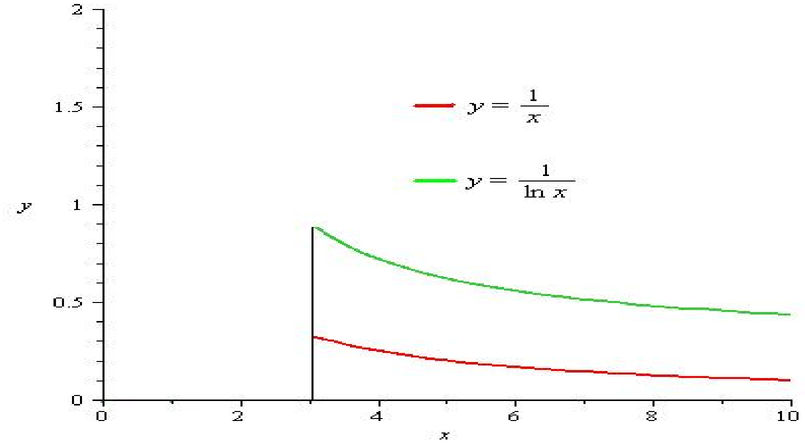

Figure: The curves $y = \frac{1}{x}$ and $y = \frac{1}{\ln x}$ plotted together for $x \geqslant 3$.

Example

Determine if the integral

$$\int_3^\infty \frac{dx}{\ln x}$$

is convergent or divergent.

$\int_3^\infty \frac{1}{x}\,dx$ is divergent

Since $0 \le \frac{1}{x} \le \frac{1}{\ln x}$ for $x \geq 3$

$\int_3^\infty \frac{1}{\ln x}\,dx$ diverges by the comparison theorem.

Determine if the Integral Converges or Diverges

(a) $\displaystyle\int_{\pi}^{\infty} \frac{2+\sin x}{x^3}\,dx$

$$2+(-1) \le 2+\sin x \le 2+1$$

$$\frac{1}{x^3} \le \frac{2+\sin x}{x^3} \le \frac{3}{x^3}$$

Since $0 \le \dfrac{2+\sin x}{x^3} \le \dfrac{3}{x^3}$ and

$\displaystyle\int_{\pi}^{\infty} \frac{3}{x^3}\,dx$ converges, $\displaystyle\int_{\pi}^{\infty} \frac{2+\sin x}{x^3}\,dx$

converges by the comparison theorem.

Determine if the Integral Converges or Diverges

(b) $\displaystyle\int_1^\infty \frac{\arctan x}{\sqrt{x}}\, dx$

$$\frac{\pi}{4} \le \tan^{-1} x \le \frac{\pi}{2}$$

$$\frac{\frac{\pi}{4}}{\sqrt{x}} \le \frac{\tan^{-1} x}{\sqrt{x}} \le \frac{\pi/2}{\sqrt{x}}$$

Since $\displaystyle\int_1^\infty \frac{\pi/4}{\sqrt{x}}\, dx$ diverges and $\displaystyle\frac{\frac{\pi}{4}}{\sqrt{x}} \le \frac{\tan^{-1} x}{\sqrt{x}}$ for all $x \ge 1$, $\displaystyle\int_1^\infty \frac{\tan^{-1} x}{\sqrt{x}}\, dx$ diverges by the Comparison theorem.