February 12 Math 3260 sec. 51 Spring 2020

Section 2.1: Matrix Operations

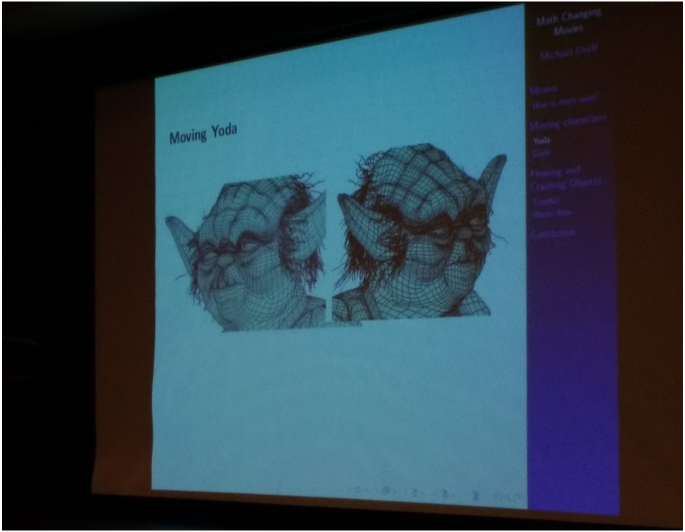
- We defined scalar multiplication and matrix addition. If A and B are $m \times n$ and c is as scalar.

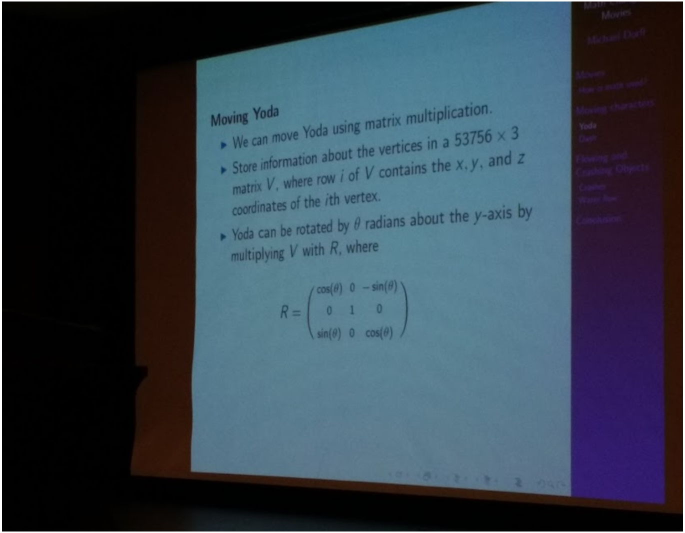
$$c[a_{ij}] = [ca_{ij}], \quad \text{and} \quad [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

- We defined matrix multiplication: If A is $m \times n$ and B is $n \times p$, then AB is defined and the product is $m \times p$.

$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p] = \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right]$$

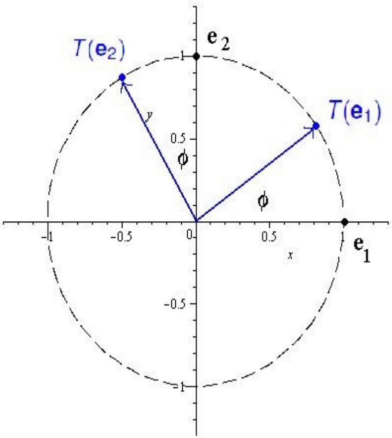
Matrix Multiplication & Graphics




Moving Yoda
We can move Yoda using matrix multiplication.
Store information about the vertices in a 53756 × 3 matrix V, where row i of V contains the x, y, and z coordinates of the ith vertex.
Yoda can be rotated by θ radians about the y-axis by multiplying V with R, where
R = (cos(θ) 0 −sin(θ) ; 0 1 0 ; sin(θ) 0 cos(θ))

A Slide from Class on February 3; Rotation in $\mathbb{R}^2$

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation transformation that rotates each point in $\mathbb{R}^2$ counter clockwise about the origin through an angle ϕ.



Using some basic trigonometry, the points on the unit circle

$$T(\mathbf{e}_1) = (\cos\phi, \sin\phi)$$

$$T(\mathbf{e}_2) = (\cos(90^\circ + \phi), \sin(90^\circ + \phi))$$

$$= (-\sin\phi, \cos\phi)$$

So $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$.

Theorem: Properties

The $m \times n$ **zero matrix** has a zero in each entry. We'll denote this matrix as O (or $O_{m,n}$ if the size is not clear from the context).

Theorem: Let A, B, and C be matrices of the same size and r and s be scalars. Then

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $A + O = A$

(iv) $r(A + B) = rA + rB$

(v) $(r + s)A = rA + sA$

(vi) $r(sA) = (rs)A$

Theorem: Properties-Matrix Product

Let A be an $m \times n$ matrix. Let r be a scalar and B and C be matrices for which the indicated sums and products are defined. Then

(i) $A(BC) = (AB)C$

(ii) $A(B + C) = AB + AC$

(iii) $(B + C)A = BA + CA$

(iv) $r(AB) = (rA)B = A(rB)$, and

(v) $I_mA = A = AI_n$

Caveats!

(1) Matrix multiplication **does not** commute! In general $AB \neq BA$.

For example, we found $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 6 \end{bmatrix}$ whereas

$$\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 11 \\ -1 & 4 \end{bmatrix}$$

(2) The zero product property **does not** hold! That is, if $AB = O$, one **cannot** conclude that one of the matrices A or B is a zero matrix.

(3) There is no *cancelation law*. That is, $AB = CB$ **does not** imply that A and C are equal.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Compute the products AB, CB, and BB.

The products

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and}$$

$$CB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}.$$

So $AB = CB$ whereas $A \neq C$. And

$$BB = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Even though B is not the zero matrix, the product BB is the zero matrix.

Matrix Powers

Positive Integer Powers: If A is square—meaning A is an $n \times n$ matrix for some $n \geq 2$, then the product AA is defined. For positive integer k, we'll define

$$A^k = AA^{k-1}.$$

Zero Power: We define $A^0 = I_n$, where I_n is the $n \times n$ identity matrix.

Transpose

Definition: Let $A = [a_{ij}]$ be an $m \times n$ matrix. The **transpose** of A is the $n \times m$ matrix denoted and defined by

$$A^T = [a_{ji}].$$

For example, if

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \quad \text{then} \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}.$$

Example

$$A = \begin{bmatrix} 5 & 5 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix}$$

Compute A^T, B^T, the transpose of the product $(AB)^T$, and the product $B^T A^T$.

$$A^T = \begin{bmatrix} 5 & -1 \\ 5 & 4 \end{bmatrix} \qquad B^T = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 3 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 5 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 35 \\ -6 & 4 & 13 \end{bmatrix}$$

2×2 2×3
match

$$(AB)^T = \begin{bmatrix} 5 & -6 \\ 5 & 4 \\ 35 & 13 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 5 & 4 \\ 35 & 13 \end{bmatrix}$$

3×2 $\quad$ 2×2

match

Theorem: Properties-Matrix Transposition

Let A and B be matrices such that the appropriate sums and products are defined, and let r be a scalar. Then

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(rA)^T = rA^T$

(iv) $(AB)^T = B^T A^T$ → It works with more than 2

$$(ABCD)^T = D^T C^T B^T A^T$$