February 13 MATH 1112 sec. 54 Spring 2019

Section 5.5: Solving Exponential and Logarithmic Equations

Base-Exponent Equality For any $a > 0$ with $a \neq 1$, and for any real numbers x and y

$$a^x = a^y \quad \text{if and only if} \quad x = y.$$

Logarithm Equality For and $a > 0$ with $a \neq 1$, and for any positive numbers x and y

$$\log_a x = \log_a y \quad \text{if and only if} \quad x = y.$$

Inverse Function For any $a > 0$ with $a \neq 1$

$$a^{\log_a x} = x \quad \text{for every} \quad x > 0$$

$$\log_a(a^x) = x \quad \text{for every real} \quad x.$$

Example

Find an exact solution[1] to the equation

$2^{x+1} = 5^x$

We solved this equation last time and found that

$$x = \frac{\ln 2}{\ln 5 - \ln 2}.$$

We used the natural log to do this, but the claim was made that any other base log could have been used.

[1] An exact solution may be a number such as $\sqrt{2}$ or $\ln(7)$ which requires a calculator to approximate as a decimal.

An Observation

To solve $2^{x+1} = 5^x$, we used the natural log. But we have choices. Use the change of base formula to show that our solution

base e → $$\frac{\ln 2}{\ln 5 - \ln 2} = \frac{\log 2}{\log 5 - \log 2}$$ ← base 10

Change of base

$$\log_a(M) = \frac{\log_b(M)}{\log_b(a)}$$

$$\ln 2 = \frac{\log 2}{\log e}$$

$$\ln 5 = \frac{\log 5}{\log e}$$

$$\frac{\ln 2}{\ln 5 - \ln 2} = \frac{\log 2/\log e}{\log 5/\log e - \log 2/\log e}$$

$$= \frac{\cancel{\frac{1}{\log e}} \log 2}{\cancel{\frac{1}{\log e}}(\log 5 - \log 2)} = \frac{\log 2}{\log 5 - \log 2}$$

Question

Jack and Diane are solving $3^{-x} = 4^{x-1}$. They arrive at the solutions

$$\text{Jack's} \quad \frac{\ln 4}{\ln 4 + \ln 3} = \frac{\log_3(4)}{\log_3(4) + \log_3(3)} \quad \text{Diane's} \quad \frac{\log_3(4)}{\log_3(4) + 1}.$$

Which of the following statements is true?

(a) Jack's answer is correct, and Diane's is incorrect.

(b) Diane's answer is correct, and Jack's is incorrect.

(c) Both answers are correct; they are the same number.

(d) Both answers are incorrect.

Log Equations & Verifying Answers

Double checking answers is always recommended. When dealing with functions whose domains are restricted, **answer verification** is critical.

Use properties of logarithms to solve the equation
$\log(x-1)+\log(x-2)=\log 12$

we want to use $\log X = \log Y \Rightarrow X = Y$

use $\log(MN) = \log M + \log N$

$\log(x-1) + \log(x-2) = \log 12$

$\log((x-1)(x-2)) = \log 12$

$(x-1)(x-2) = 12$ quadratic eqn.

$$x^2 - 3x + 2 = 12$$

$$x^2 - 3x - 10 = 0$$

$$(x-5)(x+2) = 0 \Rightarrow x - 5 = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = 5 \quad \text{or} \quad x = -2$$

The quadratic equation has solutions 5 and -2.

We have to check if these solve the original equation

$$\log(x-1) + \log(x-2) = \log 12$$

Check $x = 5$

$$\log(5-1) + \log(5-2)$$

$$\log(4) + \log(3) = \log(3 \cdot 4) = \log(12)$$

Yes 5 is a solution

Check $x = -2$ $\quad \log(-2-1) + \log(-2-2)$ undefined

-2 is not a solution.

The only solution is $x = 5$.