# February 13 Math 2306 sec. 53 Spring 2019

## Section 6: Linear Equations Theory and Terminology

Recall that an $n^{th}$ order linear IVP consists of an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called **nonhomogeneous.**

# Theorem: Existence & Uniqueness

**Theorem:** If $a_0, \ldots, a_n$ and $g$ are continuous on an interval $I$, $a_n(x) \neq 0$ for each $x$ in $I$, and $x_0$ is any point in $I$, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on $I$.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

# Homogeneous Equations

We'll consider the equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assume that each $a_i$ is continuous and $a_n$ is never zero on the interval of interest.

**Theorem:** If $y_1, y_2, \dots, y_k$ are all solutions of this homogeneous equation on an interval $I$, then the *linear combination*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution on $I$ for any choice of constants $c_1, \dots, c_k$.

This is called the **principle of superposition.**

## Corollaries

(i) If $y_1$ solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

**Big Questions:**

- Does an equation have any **nontrivial** solution(s), and
- since $y_1$ and $cy_1$ aren't truly *different* solutions, what criteria will be used to call solutions distinct?

# Linear Dependence

**Definition:** A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval $I$ if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on $I$ is said to be **linearly independent** on $I$.

The equation can always be made true by taking all c's equal to zero. If the functions are linearly dependent, then it can be done without all zero c's.

# Example: A linearly Dependent Set

The functions $f_1(x) = x^2$, $f_2(x) = 4x$, and $f_3(x) = x - x^2$ are linearly dependent on $I = (-\infty, \infty)$.

We have to show that there are numbers $c_1, c_2, c_3$ (not all zero) such that

$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all $x$ in $I$

Consider $c_1 = 1$, $c_2 = \frac{-1}{4}$ and $c_3 = 1$.

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) =$$

$$1 \cdot x^2 + \left(\frac{-1}{4}\right)(4x) + 1 \cdot (x - x^2) =$$

$$x^2 - x + x - x^2 = 0$$

At least one of $c_1, c_2, c_3$ was nonzero (all were nonzero).

Note $\quad c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$

$$c_1 x^2 + c_2(4x) + c_3(x - x^2) = 0$$

It's clear that any set where $c_1 = c_3$ and $c_2 = \frac{-1}{4} c_3$ would work.

# Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

We can show that $c_1 f_1(x) + c_2 f_2(x) = 0$ for all $x$ is only possible if $c_1 = c_2 = 0$.

Suppose $c_1 f_1(x) + c_2 f_2(x) = 0$ for all $x$

Then

$$c_1 \sin x + c_2 \cos x = 0$$

This must hold when $x = 0$. When $x = 0$, the equation is

$$c_1 \sin(0) + c_2 \cos(0) = 0$$

$$c_1 (0) + c_2 (1) = 0 \implies c_2 = 0$$

The equation must hold when $x = \pi/2$. When $x = \pi/2$, the equation is

$$c_1 \sin\left(\frac{\pi}{2}\right) + \overset{c_2}{0} \cdot \cos\left(\frac{\pi}{2}\right) = 0$$

$$c_1 (1) = 0 \quad \Longrightarrow \quad c_1 = 0$$

Both $c_1 = 0$ and $c_2 = 0$. The functions are linearly <u>independent</u>.

# Definition of Wronskian

Let $f_1, f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval $I$. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable $x$. )

# Determinants

If $A$ is a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If $A$ is a $3 \times 3$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

# Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

2 functions $\Rightarrow$ $2 \times 2$ matrix

$$W(f_1, f_2)(x) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= \sin x\,(-\sin x) - \cos x\,(\cos x)$$

$$= -\sin^2 x - \cos^2 x$$

$$= -\left(\sin^2 x + \cos^2 x\right)$$

$$= -1$$

$$W(f_1, f_2)(x) = -1$$

# Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

3 functions $\Rightarrow$ 3x3 matrix

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} x^2 & 4x & x-x^2 \\ 2x & 4 & 1-2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 4 & 1-2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1-2x \\ 2 & -2 \end{vmatrix} + (x-x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}$$

$$= x^2\left(4(-2) - 0(1-2x)\right) - 4x\left(2x(-2) - 2(1-2x)\right) + (x-x^2)\left(2x\cdot 0 - 2(4)\right)$$

$$= x^2(-8) - 4x(-4x - 2 + 4x) + (x-x^2)(-8)$$

$$= -8x^2 + 8x - 8x + 8x^2 = 0$$

$$W(f_1, f_2, f_3)(x) = 0$$

# Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval $I$. If there exists $x_0$ in $I$ such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on $I$.

If $y_1, y_2, \ldots, y_n$ are $n$ solutions of the linear homogeneous $n^{th}$ order equation on an interval $I$, then the solutions are **linearly independent** on $I$ if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for[1] each $x$ in $I$.

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[1] For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

## Determine if the functions are linearly dependent or independent:

$y_1 = x^2, \quad y_2 = x^3 \quad I = (0, \infty)$

We'll use the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= x^2(3x^2) - 2x(x^3) = 3x^4 - 2x^4 = x^4$$

$W(y_1, y_2)(x) = x^4$

Since $W \neq 0$, the functions are linearly independent.

# Fundamental Solution Set

We're still considering this equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

with the assumptions $a_n(x)\neq 0$ and $a_i(x)$ are continuous on $I$.

**Definition:** A set of functions $y_1, y_2, \dots, y_n$ is a **fundamental solution set** of the $n^{th}$ order homogeneous equation provided they

(i) are solutions of the equation,
(ii) there are $n$ of them, and
(iii) they are linearly independent.

**Theorem:** Under the assumed conditions, the equation has a fundamental solution set.