Section 6: Linear Equations Theory and Terminology

We’ll consider the equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = 0 \]

and assume that each \( a_i \) is continuous and \( a_n \) is never zero on the interval of interest.

**Theorem:** If \( y_1, y_2, \ldots, y_k \) are all solutions of this homogeneous equation on an interval \( I \), then the linear combination

\[ y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x) \]

is also a solution on \( I \) for any choice of constants \( c_1, \ldots, c_k \).

This is called the **principle of superposition**.
Linear Dependence

**Definition:** A set of functions \( f_1(x), f_2(x), \ldots, f_n(x) \) are said to be linearly dependent on an interval \( I \) if there exists a set of constants \( c_1, c_2, \ldots, c_n \) with at least one of them being nonzero such that

\[
c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all } x \text{ in } I.
\]

A set of functions that is not linearly dependent on \( I \) is said to be linearly independent on \( I \).
Examples
We saw that \( f_1(x) = x^2, \) \( f_2(x) = 4x, \) \( f_3(x) = x - x^2 \) were linearly dependent on \((\infty, \infty)\) because

\[
4x^2 - 1(4x) + 4(x - x^2) = 0 \quad \text{for all real } x
\]

At least one (in fact all) of the coefficients are nonzero. By comparison, \( f_1(x) = \sin(x), \) \( f_2(x) = \cos(x) \) are linearly independent on \((-\infty, \infty)\) because the equation

\[
c_1 \sin(x) + c_2 \cos(x) = 0 \quad \text{for all real } x
\]

is ONLY true if \( c_1 = 0 \) and \( c_2 = 0. \)

We would like a general tool to determine linear dependence/independence at least in the case that we have sufficiently differentiable functions (like the ones that solve differential equations).
Definition of Wronskian

Let $f_1, f_2, \ldots, f_n$ possess at least $n - 1$ continuous derivatives on an interval $I$. The Wronskian of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f'_1 & f'_2 & \cdots & f'_n \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$  

(Note that, in general, this Wronskian is a function of the independent variable $x$. )
Determinants

If $A$ is a $2 \times 2$ matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,
then its determinant

$\det(A) = ad - bc$.

If $A$ is a $3 \times 3$ matrix

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,
then its determinant

$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.
Determine the Wronskian of the Functions

\[ f_1(x) = \sin x, \quad f_2(x) = \cos x \]

\[
W(f_1, f_2)(x) = \begin{vmatrix}
 f_1(x) & f_2(x) \\
 f_1'(x) & f_2'(x)
\end{vmatrix}
\]

\[
\begin{vmatrix}
 \sin x & \cos x \\
 \cos x & -\sin x
\end{vmatrix}
\]

\[
= \sin x (-\sin x) - \cos x (\cos x)
\]
\[ = - \sin^2 x - \cos^2 x \]

\[ = - (\sin^2 x + \cos^2 x) = -1 \]

So \[ W(f_1, f_2)(x) = -1 \]
Determine the Wronskian of the Functions

\[ f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2 \]

3 functions, the matrix will be 3x3.

\[
W(f_1, f_2, f_3)(x) = \begin{vmatrix}
 f_1(x) & f_2(x) & f_3(x) \\
 f_1'(x) & f_2'(x) & f_3'(x) \\
 f_1''(x) & f_2''(x) & f_3''(x)
\end{vmatrix}
\]

\[
= \begin{vmatrix}
 x^2 & 4x & x - x^2 \\
 2x & 4 & 1 - 2x \\
 2 & 0 & -2
\end{vmatrix}
\]
\[
\begin{vmatrix}
1 & 1-2x & 2x & 1-2x & (x-x^2) & 2x \\
0 & -2 & 2 & -2 & (x-x^2) & 2 \\
\end{vmatrix}
\]

\[
= x^2 \left( 4(-2) - 0(1-2x) \right) - 4x \left( 2x(-2) - 2(1-2x) \right) + (x-x^2) \left( 2x(0) - 2(4) \right)
\]

\[
= x^2(-8) - 4x(4x - 2 + 4x) + (x-x^2)(-8)
\]

\[
= -8x^2 + 8x - 8x + 8x^2
\]

\[
= 0
\]
W(f_1, f_2, f_3)(x) = 0
Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n - 1$ times continuously differentiable on an interval $I$. If there exists $x_0$ in $I$ such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on $I$.

If $y_1, y_2, \ldots, y_n$ are $n$ solutions of the linear homogeneous $n^{th}$ order equation on an interval $I$, then the solutions are **linearly independent** on $I$ if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for\(^1\) each $x$ in $I$.

\(^1\)For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.
Determine if the functions are linearly dependent or independent:

\[ y_1 = e^x, \quad y_2 = e^{-2x}, \quad I = (-\infty, \infty) \]

We can use the Wronskian. (2 functions \( \rightarrow \) 2x2 matrix)

\[
W(y_1, y_2)(x) = \begin{vmatrix}
    y_1 & y_2 \\
    y_1' & y_2'
\end{vmatrix} = \begin{vmatrix}
    e^x & e^{-2x} \\
    e^x & -2e^{-2x}
\end{vmatrix}
\]
\[ = e^x (-2e^{-2x}) - e^x (e^{2x}) \]
\[ = -2e^{-x} - e^{-x} = -3e^{-x} \]

\[ W(y_1, y_2)(x) = -3e^{-x} \text{ nonzero} \]

The functions are linearly independent.
Fundamental Solution Set

We’re still considering this equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0 \]

with the assumptions \( a_n(x) \neq 0 \) and \( a_i(x) \) are continuous on \( I \).

Definition: A set of functions \( y_1, y_2, \ldots, y_n \) is a fundamental solution set of the \( n^{th} \) order homogeneous equation provided they

(i) are solutions of the equation,
(ii) there are \( n \) of them, and
(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.
General Solution of $n^{th}$ order Linear Homogeneous Equation

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the $n^{th}$ order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.
Example
Verify that \( y_1 = x^2 \) and \( y_2 = x^3 \) form a fundamental solution set of the ODE
\[
x^2 y'' - 4xy' + 6y = 0 \quad \text{on} \quad (0, \infty),
\]
and determine the general solution.

We have to show that we have the right number of functions, they are solutions, and that they are linearly independent.

For a 2nd order ODE, we have 2 functions.

Let's verify they are solutions.
\[
y_1 = x^2, \quad y_1' = 2x, \quad y_1'' = 2
\]
\[ x^2 y_1'' - 4x y_1' + 6y_1 = x^2(2) - 4x(2x) + 6x^2 = 2x^2 - 8x^2 + 6x^2 = 0 \]

\( y_1 \) solves the ODE.

\[ y_2 = x^3, \quad y_2' = 3x^2, \quad y_2'' = 6x \]

\[ x^2 y_2'' - 4x y_2' + 6y_2 = x^2(6x) - 4x(3x^2) + 6x^3 = 6x^3 - 12x^3 + 6x^3 = 0 \]

So \( y_2 \) is also a solution.

We'll use the Wronskian to show linear
The Wronskian is non-zero, so $y_1$ and $y_2$ are linearly independent. We have a fundamental solution set.

The general solution is

\[ y = C_1 x^2 + C_2 x^3. \]
Nonhomogeneous Equations

Now we will consider the equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \]

where \( g \) is not the zero function. We’ll continue to assume that \( a_n \) doesn’t vanish and that \( a_i \) and \( g \) are continuous.

The associated homogeneous equation is

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0. \]
Theorem: General Solution of Nonhomogeneous Equation

Let $y_p$ be any solution of the nonhomogeneous equation, and let $y_1, y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)
Another Superposition Principle (for nonhomogeneous eqns.)

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be $k$ particular solutions to the nonhomogeneous linear equations

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g_i(x)$$

for $i = 1, \ldots, k$. Assume the domain of definition for all $k$ equations is a common interval $I$.

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x) \frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$
Example \( x^2 y'' - 4xy' + 6y = 36 - 14x \)

We will construct the general solution by considering sub-problems.

(a) **Part 1** Verify that

\[ y_{p_1} = 6 \quad \text{solves} \quad x^2 y'' - 4xy' + 6y = 36. \]

\[ y_{p_1}' = 0 \]

\[ y_{p_1}'' = 0 \]

\[ x^2 (6) - 4x(6) + 6(6) = 36 \]
Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) **Part 2** Verify that

$$y_{p2} = -7x$$ solves $x^2y'' - 4xy' + 6y = -14x$.

$$y_{p2}' = -7$$

$$y_{p2}'' = 0$$

$$x^2y_{p2}'' - 4xy_{p2}' + 6y_{p2} =$$

$$x^2(-7) - 4x(-7) + 6(-7x) =$$

$$-28x + 28x = -14x$$
Example \( x^2y'' - 4xy' + 6y = 36 - 14x \)

(c) **Part 3** We already know that \( y_1 = x^2 \) and \( y_2 = x^3 \) is a fundamental solution set of

\[
x^2y'' - 4xy' + 6y = 0.
\]

Use this along with results (a) and (b) to write the general solution of \( x^2y'' - 4xy' + 6y = 36 - 14x \).

By the principle of superposition,

\[
y_p = y_{p1} + y_{p2} = 6 - 7x
\]

The complementary solution is \( y_c = c_1 x^2 + c_2 x^3 \).

The general solution is \( y = y_c + y_p \).
Solve the IVP

\[ x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5 \]

The general solution is

\[ y = c_1x^2 + c_2x^3 + 6 - 7x \]

\[ y' = 2c_1x + 3c_2x^2 - 7 \]

\[ y(1) = c_11^2 + c_21^3 + 6 - 7 \cdot 1 = 0 \quad \Rightarrow \quad c_1 + c_2 - 1 = 0 \]

\[ y'(1) = 2c_1 \cdot 1 + 3c_2 \cdot 1^2 - 7 = -5 \quad \Rightarrow \quad 2c_1 + 3c_2 - 7 = -5 \]
\[ c_1 + c_2 = 1 \]
\[ 2c_1 + 3c_2 = 2 \]
\[ \text{ subtract } (2c_1 + 3c_2 = 2) \]
\[ -c_2 = 0 \]

\[ c_1 = 1 - c_2 = 1 - 0 = 1 \]

The solution to the IVP is
\[ y = x^2 + 6 - 7x \]