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Section 6: Linear Equations Theory and Terminology

We'll consider the equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

and assume that each a_i is continuous and a_n is never zero on the interval of interest.

Theorem: If $y_1, y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x)=c_1y_1(x)+c_2y_2(x)+\cdots+c_ky_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

This is called the **principle of superposition**.

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \dots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

Examples

We saw that $f_1(x) = x^2$, $f_2(x) = 4x$, $f_3(x) = x - x^2$ were linearly dependent on $(-\infty, \infty)$ because

$$4x^2 - 1(4x) + 4(x - x^2) = 0 \quad \text{for all real } x$$

At least one (in fact all) of the coefficients are nonzero. By comparison, $f_1(x) = \sin(x)$, $f_2(x) = \cos(x)$ are linearly independent on $(-\infty, \infty)$ because the equation

$$c_1 \sin(x) + c_2 \cos(x) = 0 \quad \text{for all real } x$$

is ONLY true if $c_1 = 0$ and $c_2 = 0$.

We would like a general tool to determine linear dependence/independence at least in the case that we have sufficiently differentiable functions (like the ones that solve differential equations).

Definition of Wronskian

Let f_1, $f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Determinants

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If A is a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

2 functions, 2x2 matrix.

$$W(f_1, f_2)(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= \sin x(-\sin x) - \cos x(\cos x)$$

$$= -\sin^2 x - \cos^2 x$$

$$= -(\sin^2 x + \cos^2 x) = -1$$

So $\quad W(f_1, f_2)(x) = -1 .$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

3 functions, the matrix will be 3×3.

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & 4x & x - x^2 \\ 2x & 4 & 1-2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 4 & 1-2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1-2x \\ 2 & -2 \end{vmatrix} + (x-x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}$$

$$= x^2\big(4(-2) - 0(1-2x)\big) - 4x\big(2x(-2) - 2(1-2x)\big) + (x-x^2)\big(2x(0) - 2(4)\big)$$

$$= x^2(-8) - 4x(-\cancel{4x} - 2 + \cancel{4x}) + (x-x^2)(-8)$$

$$= -8x^2 + 8x - 8x + 8x^2$$

$$= 0$$

$$W(f_1, f_2, f_3)(x) = 0$$

Theorem (a test for linear independence)

Let $f_1, f_2, \dots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \dots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

If $y_1, y_2, \dots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \dots, y_n)(x) \neq 0$ for[1] each x in I.

[1] For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Determine if the functions are linearly dependent or independent:

$$y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty)$$

We can use the Wronskian. (2 fncts → 2x2 matrix)

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix}$$

$$= e^{x}(-2e^{-2x}) - e^{x}(e^{-2x})$$

$$= -2e^{-x} - e^{-x} \qquad = -3e^{-x}$$

$$W(y_1, y_2)(x) = -3e^{-x} \qquad \text{nonzero}$$

The functions are linearly independent.

Fundamental Solution Set

We're still considering this equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

with the assumptions $a_n(x)\neq 0$ and $a_i(x)$ are continuous on I.

Definition: A set of functions $y_1, y_2, \dots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,
(ii) there are n of them, and
(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of n^{th} order Linear Homogeneous Equation

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = x^2$ and $y_2 = x^3$ form a fundamental solution set of the ODE

$$x^2y'' - 4xy' + 6y = 0 \quad \text{on} \quad (0, \infty),$$

and determine the general solution.

We have to show that we have the right number of functions, they are solutions, and that they are linearly independent.

2^{nd} order ODE, we have 2 functions.

Let's verify they are solutions.

$y_1 = x^2$, $y_1' = 2x$, $y_1'' = 2$

$$x^2 y_1'' - 4x y_1' + 6y_1 = x^2(2) - 4x(2x) + 6x^2$$

$$= 2x^2 - 8x^2 + 6x^2 = 0$$

y_1 solves the ODE.

$y_2 = x^3, \quad y_2' = 3x^2, \quad y_2'' = 6x$

$$x^2 y_2'' - 4x y_2' + 6y_2 = x^2(6x) - 4x(3x^2) + 6x^3$$

$$= 6x^3 - 12x^3 + 6x^3 = 0$$

So y_2 is also a solution.

We'll use the Wronskian to show linear

independence.

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= x^2(3x^2) - 2x(x^3) = 3x^4 - 2x^4 = x^4$$

The Wronskian is nonzero, so y_1 and y_2 are linearly independent. We have a fundamental solution set.

The general solution is

$$y = C_1 x^2 + C_2 x^3.$$

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{y_c} + y_p(x)$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)

Another Superposition Principle (for nonhomogeneous eqns.)

Let y_{p_1}, y_{p_2}, ..., y_{p_k} be k particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

for $i = 1, \ldots, k$. Assume the domain of definition for all k equations is a common interval I.

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)\frac{d^ny}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

We will construct the general solution by considering sub-problems.

(a) **Part 1** Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

$y_{p_1}' = 0$

$y_{p_1}'' = 0$

$x^2 y_{p_1}'' - 4x y_{p_1}' + 6 y_{p_1} =$

$x^2(0) - 4x(0) + 6(6) = 36$ ✓

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) **Part 2** Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

$y_{p_2}' = -7$

$y_{p_2}'' = 0$

$x^2 y_{p_2}'' - 4x y_{p_2}' + 6y_{p_2} =$

$x^2(0) - 4x(-7) + 6(-7x) =$

$28x - 42x = -14x$ ✓

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) **Part 3** We already know that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0. \leftarrow \text{associated homogeneous eqn.}$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

By the principle of superposition

$$y_p = y_{p_1} + y_{p_2} = 6 - 7x$$

The complementary solution is $y_c = c_1x^2 + c_2x^3$.

The general solution is $y = c_1x^2 + c_2x^3 + 6 - 7x$.

$y_c + y_p$

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

The general solution is

$$y = c_1x^2 + c_2x^3 + 6 - 7x$$

$$y' = 2c_1x + 3c_2x^2 - 7$$

$$y(1) = c_1 1^2 + c_2 \cdot 1^3 + 6 - 7 \cdot 1 = 0 \qquad c_1 + c_2 - 1 = 0$$

$$y'(1) = 2c_1 \cdot 1 + 3c_2 \cdot 1^2 - 7 = -5 \Rightarrow 2c_1 + 3c_2 - 7 = -5$$

$$c_1 + c_2 = 1 \qquad\qquad 2c_1 + 2c_2 = 2$$

$$2c_1 + 3c_2 = 2 \qquad -(2c_1 + 3c_2 = 2)$$

$$-c_2 = 0$$

$$c_1 = 1 - c_2 = 1 - 0 = 1$$

The solution to the IVP is

$$\boxed{y = x^2 + 6 - 7x}$$