Section 6: Linear Equations Theory and Terminology

We’ll consider the equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = 0 \]

and assume that each \( a_i \) is continuous and \( a_n \) is never zero on the interval of interest.

**Theorem:** If \( y_1, y_2, \ldots, y_k \) are all solutions of this homogeneous equation on an interval \( I \), then the linear combination

\[ y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x) \]

is also a solution on \( I \) for any choice of constants \( c_1, \ldots, c_k \).

This is called the **principle of superposition**.
Linear Dependence

Definition: A set of functions $f_1(x)$, $f_2(x)$, \ldots, $f_n(x)$ are said to be linearly dependent on an interval $I$ if there exists a set of constants $c_1$, $c_2$, \ldots, $c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all $x$ in $I$.

A set of functions that is not linearly dependent on $I$ is said to be linearly independent on $I$. 
Examples

We saw that \( f_1(x) = x^2 \), \( f_2(x) = 4x \), \( f_3(x) = x - x^2 \) were linearly dependent on \((\infty, \infty)\) because

\[
4x^2 - 1(4x) + 4(x - x^2) = 0 \quad \text{for all real } x
\]

At least one (in fact all) of the coefficients are nonzero. By comparison, \( f_1(x) = \sin(x) \), \( f_2(x) = \cos(x) \) are linearly independent on \((-\infty, \infty)\) because the equation

\[
c_1 \sin(x) + c_2 \cos(x) = 0 \quad \text{for all real } x
\]

is ONLY true if \( c_1 = 0 \) and \( c_2 = 0 \).

We would like a general tool to determine linear dependence/independence at least in the case that we have sufficiently differentiable functions (like the ones that solve differential equations).
Definition of Wronskian

Let $f_1, f_2, \ldots, f_n$ possess at least $n - 1$ continuous derivatives on an interval $I$. The **Wronskian** of this set of functions is the determinant

\[
W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix}
  f_1 & f_2 & \cdots & f_n \\
  f'_1 & f'_2 & \cdots & f'_n \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} 
\end{vmatrix}.
\]

(Note that, in general, this Wronskian is a function of the independent variable $x$. )
Determinants

If $A$ is a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\text{det}(A) = ad - bc.$$ 

If $A$ is a $3 \times 3$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\text{det}(A) = a_{11} \text{det} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \text{det} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \text{det} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$
Determine the Wronskian of the Functions

\[ f_1(x) = \sin x, \quad f_2(x) = \cos x \]

2 functions \( \rightarrow \) 2x2 matrix

\[
W(f_1, f_2)(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f'_1(x) & f'_2(x) \end{vmatrix}
\]

\[
= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}
\]

\[
= \sin x (-\sin x) - \cos x (\cos x)
\]
\[ = -\sin^2 x - \cos^2 x \]

\[ = - \left( \sin^2 x + \cos^2 x \right) \]

\[ = -1 \]

\[ W(f_1, f_2)(x) = -1. \]
Determine the Wronskian of the Functions

\[ f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2 \]

3 functions \rightarrow 3 \times 3 \text{ matrix}

\[
W(f_1, f_2, f_3)(x) = \begin{vmatrix}
  f_1(x) & f_2(x) & f_3(x) \\
  f_1'(x) & f_2'(x) & f_3'(x) \\
  f_1''(x) & f_2''(x) & f_3''(x)
\end{vmatrix}
\]

\[
= \begin{vmatrix}
  x^2 & 4x & x - x^2 \\
  2x & 4 & 1 - 2x \\
  2 & 0 & -2
\end{vmatrix}
\]
\[
\begin{vmatrix}
4 & 1-2x \\
0 & -2
\end{vmatrix}
\begin{vmatrix}
-4x \\
2 -2
\end{vmatrix}
+ (x-x^2)
\begin{vmatrix}
2x & 4 \\
2 & 0
\end{vmatrix}
\]

\[
=x^2\left(4(-2)-0(1-2x)\right) - 4x \left(2x(-2)-2(1-2x)\right) + (x-x^2)(2x(0)-2(4))
\]

\[
=x^2(-8) - 4x(-4x - 2 + 4x) + (x-x^2)(-8)
\]

\[
=-8x^2 + 8x - 8x + 8x^2
\]

\[
=0
\]

\[
W(f_1, f_2, f_3) (x) = 0
\]
Theorem (a test for linear independence)

Let \( f_1, f_2, \ldots, f_n \) be \( n - 1 \) times continuously differentiable on an interval \( I \). If there exists \( x_0 \) in \( I \) such that \( W(f_1, f_2, \ldots, f_n)(x_0) \neq 0 \), then the functions are linearly independent on \( I \).

If \( y_1, y_2, \ldots, y_n \) are \( n \) solutions of the linear homogeneous \( n^{th} \) order equation on an interval \( I \), then the solutions are linearly independent on \( I \) if and only if \( W(y_1, y_2, \ldots, y_n)(x) \neq 0 \) for each \( x \) in \( I \).

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\(^1\)For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.
Determine if the functions are linearly dependent or independent:

\[ y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty) \]

We can use the Wronskian.

\[
W(y_1, y_2)(x) = \begin{vmatrix}
    y_1 & y_2 \\
    y_1' & y_2'
\end{vmatrix}
\]

\[
= \begin{vmatrix}
    e^x & e^{-2x} \\
    e^x & -2e^{-2x}
\end{vmatrix}
\]
\[ e^x (-2e^{-2x}) - e^x (e^{-2x}) = -2e^{-x} - e^{-x} = -3e^{-x} \]

\[ W(y_1, y_2)(x) = -3e^{-x} \quad \text{nonzero} \]

The functions are linearly independent.
Fundamental Solution Set

We’re still considering this equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = 0 \]

with the assumptions \( a_n(x) \neq 0 \) and \( a_i(x) \) are continuous on \( I \).

Definition: A set of functions \( y_1, y_2, \ldots, y_n \) is a fundamental solution set of the \( n^{th} \) order homogeneous equation provided they

(i) are solutions of the equation,
(ii) there are \( n \) of them, and
(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.
General Solution of $n^{th}$ order Linear Homogeneous Equation

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the $n^{th}$ order linear homogeneous equation. Then the \textbf{general solution} of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.
Example

Verify that \( y_1 = x^2 \) and \( y_2 = x^3 \) form a fundamental solution set of the ODE

\[
x^2 y'' - 4xy' + 6y = 0 \quad \text{on} \quad (0, \infty),
\]

and determine the general solution.

We need to show that \( y_1, y_2 \) are solutions and that they are linearly independent.

Show \( y_1 \) is a solution:

\[
y_1 = x^2, \quad y_1' = 2x, \quad y_1'' = 2
\]

\[
x^2 y_1'' - 4xy_1' + 6y_1 = x^2(2) - 4x(2x) + 6(x^2)
\]

\[
= 2x^2 - 8x^2 + 6x^2 = 0
\]
$y_1$ is a solution.

Show $y_2$ solves the ODE.

$y_2 = x^3, \quad y_2' = 3x^2, \quad y_2'' = 6x$

$x^2 y_2'' - xy_2' + 6y_2 = x^2(6x) - 4x(3x^2) + 6(x^3)$

$= 6x^3 - 12x^3 + 6x^3$

$= 0$

So $y_2$ solves it as well.

We can use the Wronskian to show they
are linearly independent,

\[ W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \]

\[ = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^2(3x^2) - 2x(x^3) \]

\[ = 3x^4 - 2x^4 = x^4 \]

\[ W(y_1, y_2) = x^4 \neq 0 \]

They are lin. independent. We have a fundamental solution set. The general solution

\[ y = C_1 x^2 + C_2 x^3. \]
Nonhomogeneous Equations

Now we will consider the equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \]

where \( g \) is not the zero function. We’ll continue to assume that \( a_n \) doesn’t vanish and that \( a_i \) and \( g \) are continuous.

The associated homogeneous equation is

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0. \]
Theorem: General Solution of Nonhomogeneous Equation

Let $y_p$ be any solution of the nonhomogeneous equation, and let $y_1$, $y_2$, $\ldots$, $y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where $c_1$, $c_2$, $\ldots$, $c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)
Another Superposition Principle (for nonhomogeneous eqns.)

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be $k$ particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

for $i = 1, \ldots, k$. Assume the domain of definition for all $k$ equations is a common interval $I$.

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$
Example \( x^2y'' - 4xy' + 6y = 36 - 14x \)

We will construct the general solution by considering sub-problems.

(a) **Part 1** Verify that

\[
y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.
\]

\[
y_{p_1}' = 0 \\
y_{p_1}'' = 0
\]

\[
x^2(0) - 4x(0) + 6 \cdot 6 = 36
\]

So, \( y_{p_1} \) is a particular solution.
Example $x^2 y'' - 4xy' + 6y = 36 - 14x$

(b) **Part 2** Verify that

$$y_{p_2} = -7x$$

solves

$$x^2 y'' - 4xy' + 6y = -14x.$$ 

$$y_{p_2}' = -7$$

$$y_{p_2}'' = 0$$

$$x^2 y_{p_2}'' - 4x y_{p_2}' + 6 y_{p_2} =$$

$$x^2 (-7) - 4x (-7) + 6 (-7x) =$$

$$28x - 42x = -14x$$

$y_{p_2}$ is a particular solution.
Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) Part 3 We already know that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$ 

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

$$y_c = C_1y_1 + C_2y_2 = C_1x^2 + C_2x^3$$

By the principle of superposition $y_p = y_{p1} + y_{p2} = 6 - 7x$

The general solution to the nonhomogeneous equation is

$$y = y_c + y_p = C_1x^2 + C_2x^3 + 6 - 7x.$$
Solve the IVP

\[ x^2 y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = 5 \]

The general solution to the ODE is

\[ y = c_1 x^2 + c_2 x^3 + 6 - 7x \]

So \[ y' = 2c_1 x + 3c_2 x^2 - 7 \]

Applying the I.C.,

\[ y(1) = c_1 (1)^2 + c_2 (1)^3 + 6 - 7 \cdot 1 = 0 \]

\[ c_1 + c_2 - 1 = 0 \]

\[ c_1 + c_2 = 1 \]
\[ y'(1) = 2c_1 \cdot 1 + 3c_2(1)^2 - 7 = 5 \]
\[ 2c_1 + 3c_2 - 7 = 5 \]
\[ 2c_1 + 3c_2 = 12 \]

\[ c_1 + c_2 = 1 \Rightarrow 2c_1 + 2c_2 = 2 \]
\[ 2c_1 + 3c_2 = 12 \]
\[ -c_2 = -10 \]
\[ c_2 = 10 \]

\[ c_1 = 1 - c_2 = 0.9 \]

The solution to the IVP is
\[ y = -9x^2 + 10x^3 + 6 - 7x \]