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Section 6: Linear Equations Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

to solve subject to conditions

$$y(x_0)=y_0,\quad y'(x_0)=y_1,\quad \ldots,\quad y^{(n-1)}(x_0)=y_{n-1}.$$

The problem is called **homogeneous** if $g(x)\equiv 0$. Otherwise it is called **nonhomogeneous.**

Theorem: Existence & Uniqueness

Theorem: If $a_0, \dots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \dots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

Homogeneous Equations

We'll consider the equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assume that each a_i is continuous and a_n is never zero on the interval of interest.

Theorem: If $y_1, y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

This is called the **principle of superposition.**

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct?

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

Note that choosing all c's equal to zero will always make that sum zero. Linear dependence means it can be done with at least one $c \neq 0$

Example: A linearly Dependent Set

The functions $f_1(x) = x^2$, $f_2(x) = 4x$, and $f_3(x) = x - x^2$ are linearly dependent on $I = (-\infty, \infty)$.

We need to show that there are numbers c_1, c_2, c_3 not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f(3) = 0 \quad \text{for all } x \text{ in } I$$

Consider $c_1 = 1$, $c_2 = \frac{-1}{4}$, $c_3 = 1$. Then at least one is nonzero (they all are nonzero!).

Note that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) =$$

$$1x^2 + \left(\frac{-1}{4}\right)4x + 1(x - x^2) =$$

$$x^2 - x + x - x^2 = 0$$ for all x in $(-\infty, \infty)$.

* Note $f_1(x) = x^2$, $f_2(x) = 4x$, $f_3(x) = x - x^2$

Observe $c_1 x^2 + c_2(4x) + c_3(x - x^2) = 0$

It's clear that this will hold whenever

$c_1 = c_3$ and $c_2 = \frac{-1}{4} c_3$

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

We can show that if $c_1 f_1(x) + c_2 f_2(x) = 0$ for all x in I, then it must be that $c_1 = c_2 = 0$.

Suppose $c_1 f_1(x) + c_2 f_2(x) = 0$ for all real x

$$c_1 \sin x + c_2 \cos x = 0$$

This has to hold when $x = 0$. When $x=0$, the equation is

$$c_1 \sin(0) + c_2 \cos(0) = 0$$

$$c_1(0) + c_2(1) = 0 \implies c_2 = 0$$

The equation also has to hold when $x = \pi/2$.

when $x = \frac{\pi}{2}$, the equation is

c_2

$$c_1 \sin\left(\frac{\pi}{2}\right) + 0 \cdot \cos\frac{\pi}{2} = 0$$

$$c_1(1) = 0 \implies c_1 = 0$$

Both c's must be zero. f_1 and f_2 are linearly independent.

Definition of Wronskian

Let f_1, $f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Determinants

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If A is a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

2 functions, the matrix will be 2x2.

$$f_1'(x) = \cos x \qquad f_2'(x) = -\sin x$$

$$W(f_1, f_2)(x) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= \sin x(-\sin x) - \cos x(\cos x)$$

$$= -\sin^2 x - \cos^2 x$$

$$= -(\sin^2 x + \cos^2 x) = -1$$

$$W(f_1, f_2)(x) = -1$$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

3 functions, the matrix will be 3×3

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} x^2 & 4x & x-x^2 \\ 2x & 4 & 1-2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 4 & 1-2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1-2x \\ 2 & -2 \end{vmatrix} + (x-x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}$$

$$= x^2\left(4(-2)-0(1-2x)\right)-4x\left(2x(-2)-2(1-2x)\right)+(x-x^2)\left(2x\cdot 0-2\cdot 4\right)$$

$$= x^2(-8)-4x\left(-4x-2+4x\right)+(x-x^2)(-8)$$

$$= -8x^2+8x-8x+8x^2$$

$$= 0$$

$$W(f_1,f_2,f_3)(x) = 0$$

Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

If $y_1, y_2, \ldots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for[1] each x in I.

[1] For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Determine if the functions are linearly dependent or independent:

$$y_1 = x^2, \quad y_2 = x^3 \quad I = (0, \infty)$$

We can use the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= x^2(3x^2) - (2x)(x^3) = 3x^4 - 2x^4 = x^4$$

$W(y_1, y_2)(x) = x^4$

This is nonzero for every x in $(0,\infty)$.

$W \neq 0$ the functions are linearly independent.

Fundamental Solution Set

We're still considering this equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

with the assumptions $a_n(x) \neq 0$ and $a_i(x)$ are continuous on I.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.