February 13 Math 3260 sec. 55 Spring 2018

Section Section 1.9: The Matrix for a Linear Transformation

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n \times n$ identity matrix I_n. That is,

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]\,.$$

The matrix A is called the **standard matrix** for the linear transformation T.

One to One, Onto

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of at least one **x** in $\mathbb{R}^n$—i.e. if the range of T is all of the codomain.

If T is onto w/ standard matrix A

then $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^m$.

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each **b** in $\mathbb{R}^m$ is the image of **at most one x** in $\mathbb{R}^n$.

If T is one to one

$T(\vec{x}) = T(\vec{y})$ if and only if $\vec{x} = \vec{y}$.

When $A\vec{x} = \vec{b}$ is consistent there are no free variables.

Determine if the transformation is one to one, onto, neither or both.

$\swarrow A$

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}\mathbf{x}.$$

$T: \mathbb{R}^3 \to \mathbb{R}^2$

Is T onto? Is $A\vec{x} = \vec{b}$ consistent for each $\vec{b}$ in $\mathbb{R}^2$?

Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ in $\mathbb{R}^2$. An augmented matrix for the system is

$$\begin{bmatrix} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{bmatrix}$$

This is already an rref. Column 4 is not a pivot column for any b_1, b_2. The system is consistent for all $\vec{b}$ in $\mathbb{R}^2$.

Hence T is onto.

Is T one to one? If $A\vec{x} = \vec{b}$ is consistent, is there exactly one solution?

$$\begin{bmatrix} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{bmatrix}$$

Solutions would satisfy $x_1 = b_1 - 2x_3$
$x_2 = b_2 - 3x_3$
x_3 - free

So $A\vec{x} = \vec{b}$ has infinitely many solutions.

T is not one to one.

Some Theorems

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if the homogeneous equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Recall, for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$

$T(\vec{0}) = \vec{0}$.

If T is one to one, and $T(\vec{x}) = \vec{0}$ for some $\vec{x}$, then $\vec{x} = \vec{0}$.

Conversely, suppose $T(\vec{x}) = \vec{0}$ has only the trivial solution. Suppose vectors $\vec{x}$ and $\vec{y}$

satisfy $T(\vec{x}) = T(\vec{y})$.

Using the fact that T is linear

$$T(\vec{x}) - T(\vec{y}) = \vec{0}$$

$$T(\vec{x} - \vec{y}) = \vec{0}$$

But then $\vec{x} - \vec{y} = \vec{0}$! Hence $\vec{x} = \vec{y}$.

That is, T is one to one.

Some Theorems

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(i) T is onto if and only if the columns of A span $\mathbb{R}^m$, and

(ii) T is one to one if and only if the columns of A are linearly independent.

Example

Let $T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_2)$. Verify that T is one to one. Is T onto?

Let's build the standard matrix A.

$$T(\vec{e}_1) = T(1,0) = (1, 2\cdot1 - 0, 3\cdot0) = (1,2,0)$$

$$T(\vec{e}_2) = T(0,1) = (0, 2\cdot0 - 1, 3\cdot1) = (0,-1,3)$$

$$A = \left[T(\vec{e}_1) \quad T(\vec{e}_2)\right] = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$$

To show that T is one to one, we can show that the columns of A are lin. independent.

$$A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The columns are linearly independent.

Is T onto?

Is $A\vec{x} = \vec{b}$ always consistent?

$$\begin{bmatrix} 1 & 0 & b_1 \\ 2 & -1 & b_2 \\ 0 & 3 & b_3 \end{bmatrix} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 0 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 3 & b_3 \end{bmatrix}$$

$$3R_2 + R_3 \to R_3 \qquad \begin{bmatrix} 1 & 0 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + 3b_2 - 6b_1 \end{bmatrix}$$

$\vec{b}$ is only in the range of T if

$$b_3 + 3b_2 - 6b_1 = 0.$$

So T is not onto.

Section 2.1: Matrix Operations

Recall the convenient notaton for a matrix A

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Here each column is a vector $\mathbf{a}_j$ in $\mathbb{R}^m$. We'll use the additional convenient notation to refer to A by entries

$$A = [a_{ij}].$$

a_{ij} is the entry in **row** i and **column** j.
Main Diagonal: The main diagonal consist of the entries a_{ii}. A **diagonal matrix** is a square matrix $m = n$ for which all entries **not** on the main diagonal are zero.

Scalar Multiplication, Matrix Addition, & Equality

Scalar Multiplication: For $m \times n$ matrix $A = [a_{ij}]$ and scalar c

$$cA = [ca_{ij}].$$

Matrix Addition: For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices is only defined if they are of the same size.

Matrix Equality: Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal (i.e. $A = B$) provided

$$a_{ij} = b_{ij} \quad \text{for every} \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n.$$

Example

$$A=\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B=\begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C=\begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Evaluate each expression or state why it fails to exist.

(a) 3B

$$=\begin{bmatrix} 3(-2) & 3(4) \\ 3(7) & 3(0) \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ 21 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

(b) A + B $= \begin{bmatrix} 1+(-2) & -3+4 \\ -2+7 & 2+0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 5 & 2 \end{bmatrix}$

Defined

A, B both 2x2

(c) C + A

A is 2x2, C is 2x3

A+C is not defined since they are not the same size.

Theorem: Properties

The $m \times n$ **zero matrix** has a zero in each entry. We'll denote this matrix as O (or $O_{m,n}$ if the size is not clear from the context).

Theorem: Let A, B, and C be matrices of the same size and r and s be scalars. Then

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $A + O = A$

(iv) $r(A + B) = rA + rB$

(v) $(r + s)A = rA + sA$

(vi) $r(sA) = (rs)A = (sr)A$
$= s(rA)$

Matrix Multiplication

We know that for any $m \times n$ matrix A, the operation **"multiply vectors in $\mathbb{R}^n$ by A"** defines a linear transformation (from $\mathbb{R}^n$ to $\mathbb{R}^m$).

We wish to define matrix multiplication in such a way as to correspond to **function composition.** Thus if

$$S(\mathbf{x}) = B\mathbf{x}, \quad \text{and} \quad T(\mathbf{v}) = A\mathbf{v},$$

then

$$(T \circ S)(\mathbf{x}) = T(S(\mathbf{x})) = A(B\mathbf{x}) = (AB)\mathbf{x}.$$

Matrix Multiplication

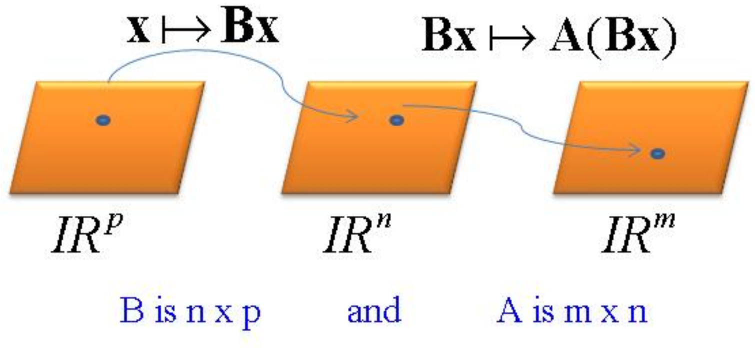

Figure: Composition requires the number of rows of B match the number of columns of A. Otherwise the product is **not defined**.