Section 1.9: The Matrix for a Linear Transformation

**Theorem:** Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation. There exists a unique \( m \times n \) matrix \( A \) such that

\[
T(x) = Ax \quad \text{for every} \quad x \in \mathbb{R}^n.
\]

Moreover, the \( j^{th} \) column of the matrix \( A \) is the vector \( T(e_j) \), where \( e_j \) is the \( j^{th} \) column of the \( n \times n \) identity matrix \( I_n \). That is,

\[
A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}.
\]

The matrix \( A \) is called the **standard matrix** for the linear transformation \( T \).
One to One, Onto

**Definition:** A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $b$ in $\mathbb{R}^m$ is the image of at least one $x$ in $\mathbb{R}^n$—i.e. if the range of $T$ is all of the codomain.

If $T$ is onto, all standard matrix $A$

then $A\hat{x} = \hat{b}$ is consistent for every $\hat{b}$ in $\mathbb{R}^m$.

**Definition:** A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one to one** if each $b$ in $\mathbb{R}^m$ is the image of **at most one** $x$ in $\mathbb{R}^n$.

If $T$ is one to one

$T(\hat{x}) = T(\hat{y})$ if and only if $\hat{x} = \hat{y}$.

When $A\hat{x} = \hat{b}$ is consistent then one no free variables.
Determine if the transformation is one to one, onto, neither or both.

\[ T(x) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} x. \]

Let \( A \) be a matrix.

\[ \begin{align*}
T: \mathbb{R}^3 & \rightarrow \mathbb{R}^2 \\
T(x) & = b \in \mathbb{R}^2.
\end{align*} \]

Is \( T \) onto? Is \( Ax = b \) consistent for each \( b \) in \( \mathbb{R}^2 \)?

Let \( b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \) in \( \mathbb{R}^2 \). An augmented matrix for the system is

\[ \begin{bmatrix} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{bmatrix} \]
This is already on ref. Column 4 is not a pivot column for any \( b_1, b_2 \). The system is consistent for all \( b \) in \( \mathbb{R}^2 \).

Hence \( T \) is onto.

Is \( T \) one to one? If \( \mathbf{A} \mathbf{x} = \mathbf{b} \) is consistent, is there exactly one solution?

\[
\begin{bmatrix}
1 & 0 & 2 & b_1 \\
0 & 1 & 3 & b_2
\end{bmatrix}
\]
Solutions would satisfy

\[ x_1 = b_1 - 2x_3 \]
\[ x_2 = b_2 - 3x_3 \]
\[ x_3 \text{ - free} \]

so \( A\hat{x} = \hat{b} \) has infinitely many solutions.

\( T \) is not one to one.
Some Theorems

**Theorem:** Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then $T$ is one to one if and only if the homogeneous equation $T(x) = 0$ has only the trivial solution.

Recall, for any linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

$$T(\tilde{0}) = 0.$$  

If $T$ is one to one, and $T(\tilde{x}) = \tilde{0}$ for some $\tilde{x}$, then $\tilde{x} = \tilde{0}$.

Conversely, suppose $T(\tilde{x}) = \tilde{0}$ has only the
trivial solution. Suppose vectors $\vec{x}$ and $\vec{y}$ satisfy

\[ T(\vec{x}) = T(\vec{y}) \, . \]

Using the fact that $T$ is linear

\[ T(\vec{x} - \vec{y}) = 0 \]

But then $\vec{x} - \vec{y} = 0$! Hence $\vec{x} = \vec{y}$.

That is, $T$ is one to one.
Some Theorems

**Theorem**: Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation, and let $A$ be the standard matrix for $T$. Then

(i) $T$ is onto if and only if the columns of $A$ span $\mathbb{R}^m$, and

(ii) $T$ is one to one if and only if the columns of $A$ are linearly independent.
Example

Let \( T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_2) \). Verify that \( T \) is one to one. Is \( T \) onto?

Let's build the standard matrix \( A \).

\[
T(\mathbf{e}_1) = T(1, 0) = (1, 2 \cdot 1 - 0, 3 \cdot 0) = (1, 2, 0)
\]

\[
T(\mathbf{e}_2) = T(0, 1) = (0, 2 \cdot 0 - 1, 3 \cdot 1) = (0, -1, 3)
\]

\[
A = \begin{bmatrix}
T(\mathbf{e}_1) & T(\mathbf{e}_2)
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
2 & -1 \\
0 & 3
\end{bmatrix}
\]

To show that \( T \) is one to one, we can show
that the columns of $A$ are lin. independent.

$$A \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The columns are linearly independent.

Is $T$ onto?

Is $A\vec{x} = \vec{b}$ always consistent?

$$\begin{bmatrix} 1 & 0 & b_1 \\ 2 & -1 & b_2 \\ 0 & 3 & b_3 \end{bmatrix} \rightarrow -2R_1 + R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 3 & b_3 \end{bmatrix}$$
\[3b_2 + b_3 \rightarrow b_3 \quad \begin{bmatrix} 
1 & 0 & b_1 \\
0 & -1 & b_2 - 2b_1 \\
0 & 0 & b_3 + 3b_2 - 6b_1 
\end{bmatrix}\]

\(b_1\) is only in the range of \(T\) if

\[b_3 + 3b_2 - 6b_1 = 0\]

So \(T\) is not onto.
Section 2.1: Matrix Operations

Recall the convenient notation for a matrix $A$

$$A = [a_1 \ a_2 \ \cdots \ a_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$  

Here each column is a vector $a_j$ in $\mathbb{R}^m$. We’ll use the additional convenient notation to refer to $A$ by entries

$$A = [a_{ij}].$$

$a_{ij}$ is the entry in row $i$ and column $j$.

**Main Diagonal:** The main diagonal consist of the entries $a_{ii}$. A **diagonal matrix** is a square matrix $m = n$ for which all entries not on the main diagonal are zero.
Scalar Multiplication, Matrix Addition, & Equality

Scalar Multiplication: For $m \times n$ matrix $A = [a_{ij}]$ and scalar $c$

$$cA = [ca_{ij}].$$

Matrix Addition: For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices is only defined if they are of the same size.

Matrix Equality: Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal (i.e. $A = B$) provided

$$a_{ij} = b_{ij} \quad \text{for every} \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n.$$
Example

\[
A = \begin{bmatrix}
1 & -3 \\
-2 & 2
\end{bmatrix}, \quad B = \begin{bmatrix}
-2 & 4 \\
7 & 0
\end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix}
2 & 0 & 2 \\
1 & -4 & 6
\end{bmatrix}
\]

Evaluate each expression or state why it fails to exist.

(a) \( 3B \)

\[
= \begin{bmatrix}
3(-2) & 3(4)
\end{bmatrix} = \begin{bmatrix}
-6 & 12
\end{bmatrix} = \begin{bmatrix}
21 & 0
\end{bmatrix}
\]
\[ A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix} \]

(b) \quad A + B = \begin{bmatrix} 1 + (-2) & -3 + 4 \\ -2 + 7 & 2 + 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 5 & 2 \end{bmatrix}

Defined \quad A, B \ \text{both} \ 2 \times 2

(c) \quad C + A

A \text{ is } 2 \times 2, \quad C \text{ is } 2 \times 3

A + C \text{ is not defined since they are not the same size.}
Theorem: Properties

The $m \times n$ zero matrix has a zero in each entry. We’ll denote this matrix as $O$ (or $O_{m,n}$ if the size is not clear from the context).

**Theorem:** Let $A$, $B$, and $C$ be matrices of the same size and $r$ and $s$ be scalars. Then

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $A + O = A$

(iv) $r(A + B) = rA + rB$

(v) $(r + s)A = rA + sA$

(vi) $r(sA) = (rs)A = s(rA)$
Matrix Multiplication

We know that for any $m \times n$ matrix $A$, the operation "multiply vectors in $\mathbb{R}^n$ by $A$" defines a linear transformation (from $\mathbb{R}^n$ to $\mathbb{R}^m$).

We wish to define matrix multiplication in such a way as to correspond to function composition. Thus if

$$S(x) = Bx, \quad \text{and} \quad T(v) = Av,$$

then

$$(T \circ S)(x) = T(S(x)) = A(Bx) = (AB)x.$$
Matrix Multiplication

Figure: Composition requires the number of rows of $B$ match the number of columns of $A$. Otherwise the product is not defined.