February 13 Math 3260 sec. 56 Spring 2018

Section Section 1.9: The Matrix for a Linear Transformation

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n \times n$ identity matrix I_n. That is,

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]\,.$$

The matrix A is called the **standard matrix** for the linear transformation T.

One to One, Onto

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of at least one **x** in $\mathbb{R}^n$—i.e. if the range of T is all of the codomain. If $T(\vec{x}) = A\vec{x}$ T is onto if

$A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^m$.

Definition: A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each **b** in $\mathbb{R}^m$ is the image of **at most one x** in $\mathbb{R}^n$.

T is one to one if $T(\vec{x}) = T(\vec{y})$ if and only if $\vec{x} = \vec{y}$. If $T(\vec{x}) = A\vec{x}$, the T is one to one if $A\vec{x} = \vec{b}$ has no free variables.

Determine if the transformation is one to one, onto, neither or both.

A

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}.$$

$T: \mathbb{R}^3 \to \mathbb{R}^2$.

Is T one to one? If $A\vec{x}=\vec{b}$ is consistent, are there free variables? If $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$A\vec{x}=\vec{b}$ has augmented matrix

$$\begin{bmatrix} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{bmatrix}$$

This is an rref. 2 pivot columns w/ 3 variables so there is a free variable.

T is not one to one.

Is T onto? Is $A\vec{x} = \vec{b}$ consistent for every $\vec{b}$ in $\mathbb{R}^2$?

$$\begin{bmatrix} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{bmatrix}$$

The last column can't be a pivot column for any choice of b_1, b_2.

Hence T is onto.

Some Theorems

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if the homogeneous equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Recall that for any linear transformation

$$T(\vec{0}) = \vec{0}.$$

If T is one to one, then clearly $T(\vec{x}) = \vec{0}$ has only the trivial solution.

Suppose $T(\vec{x}) = \vec{0}$ has only the trivial solution.

Let $T(\vec{x}) = T(\vec{y})$ for some $\vec{x}, \vec{y}$ in $\mathbb{R}^n$.

Since T is linear,

$$T(\vec{x}) - T(\vec{y}) = \vec{0}$$

$$T(\vec{x} - \vec{y}) = \vec{0}$$

Since the homogeneous eqn has only the trivial solution $\vec{x} - \vec{y} = \vec{0}$. Hence $\vec{x} = \vec{y}$ and T is one to one.

Some Theorems

Theorem: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(i) T is onto if and only if the columns of A span $\mathbb{R}^m$, and

(ii) T is one to one if and only if the columns of A are linearly independent.

Example

Let $T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_2)$. Verify that T is one to one. Is T onto?

$T : \mathbb{R}^2 \to \mathbb{R}^3$ We can build the standard matrix A. $A = [T(\vec{e}_1)\ T(\vec{e}_2)]$

$$T(\vec{e}_1) = T(1,0) = (1, 2\cdot 1 - 0, 3\cdot 0) = (1,2,0)$$

$$T(\vec{e}_2) = T(0,1) = (0, 2\cdot 0 - 1, 3\cdot 1) = (0,-1,3)$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix} \qquad A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The columns of A are linearly independent.

Hence T is one to one.

To determine if T is onto, we can consider

$A\vec{x} = \vec{b}$ for arbitrary $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in $\mathbb{R}^3$,

An augment matrix is

$$\begin{bmatrix} 1 & 0 & b_1 \\ 2 & -1 & b_2 \\ 0 & 3 & b_3 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & 0 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + 3b_2 - 6b_1 \end{bmatrix}$$

T is not onto. $A\vec{x} = \vec{b}$ is only consistent

if $\quad b_3 + 3b_2 - 6b_1 = 0$.

Section 2.1: Matrix Operations

Recall the convenient notaton for a matrix A

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Here each column is a vector $\mathbf{a}_j$ in $\mathbb{R}^m$. We'll use the additional convenient notation to refer to A by entries

$$A = [a_{ij}].$$

a_{ij} is the entry in **row** i and **column** j.
Main Diagonal: The main diagonal consist of the entries a_{ii}. A **diagonal matrix** is a square matrix $m = n$ for which all entries **not** on the main diagonal are zero.

Scalar Multiplication, Matrix Addition, & Equality

Scalar Multiplication: For $m \times n$ matrix $A = [a_{ij}]$ and scalar c

$$cA = [ca_{ij}].$$

Matrix Addition: For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices is only defined if they are of the same size.

Matrix Equality: Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal (i.e. $A = B$) provided

$$a_{ij} = b_{ij} \quad \text{for every} \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n.$$

Example

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Evaluate each expression or state why it fails to exist.

(a) 3B $= \begin{bmatrix} -2(3) & 4(3) \\ 7(3) & 0(3) \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ 21 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

(b) A + B

$$= \begin{bmatrix} 1+(-2) & -3+4 \\ -2+7 & 2+0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 5 & 2 \end{bmatrix}$$

Both 2×2

(c) C + A

C is 2×3 and A is 2×2. They're not the same size, C+A is not defined.

Theorem: Properties

The $m \times n$ **zero matrix** has a zero in each entry. We'll denote this matrix as O (or $O_{m,n}$ if the size is not clear from the context).

Theorem: Let A, B, and C be matrices of the same size and r and s be scalars. Then

(i) $A + B = B + A$	(iv) $r(A + B) = rA + rB$
(ii) $(A + B) + C = A + (B + C)$	(v) $(r + s)A = rA + sA$
(iii) $A + O = A$	(vi) $r(sA) = (rs)A = (sr)A$ $= s(rA)$

Matrix Multiplication

We know that for any $m \times n$ matrix A, the operation **"multiply vectors in $\mathbb{R}^n$ by A"** defines a linear transformation (from $\mathbb{R}^n$ to $\mathbb{R}^m$).

We wish to define matrix multiplication in such a way as to correspond to **function composition.** Thus if

$$S(\mathbf{x}) = B\mathbf{x}, \quad \text{and} \quad T(\mathbf{v}) = A\mathbf{v},$$

then

$$(T \circ S)(\mathbf{x}) = T(S(\mathbf{x})) = A(B\mathbf{x}) = (AB)\mathbf{x}.$$