February 14 Math 1190 sec. 62 Spring 2017

Section 2.2: The Derivative as a Function

Definition: Let f be a function. The *derivative* of f is the function denoted f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for each x in the domain of f for which the limit exists.
f' is read as "f prime."

Remarks:

- if $f(x)$ is a function of x, then $f'(x)$ is a new function of x (called the derivative of f)
- The number $f'(c)$ (if it exists) is the slope of the curve and of the tangent line to the curve $y = f(x)$ at the point $(c, f(c))$
- $f'(c)$ is the *rate of change* of the function f at c.

Definition: A function f is said to be *differentiable* at c if $f'(c)$ exists. It is called *differentiable* on an open interval I if it is differentiable at each point in I.

Failure to be Differentiable

We saw that the domain of $f(x)=\sqrt{x-1}$ is $[1,\infty)$ whereas the domain of its derivative $f'(x)=\frac{1}{2\sqrt{x-1}}$ was $(1,\infty)$. Hence f **is not differentiable at** 1.

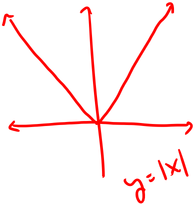
Another Example: Show that $y=|x|$ is not differentiable at zero.

Let $f(x)=|x|$, $f(0)=|0|=0$, $f(0+h)=|0+h|=|h|$

If it exists

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{|h|-0}{h}=\lim_{h\to 0}\frac{|h|}{h}$$



$$|h| = \begin{cases} h, & h \geq 0 \\ -h, & h < 0 \end{cases}$$

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h}$$

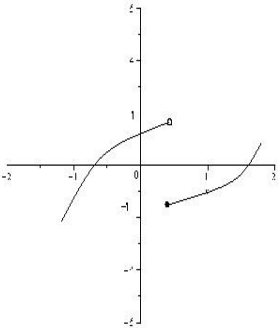
$$= \lim_{h\to 0^+} 1 = 1$$

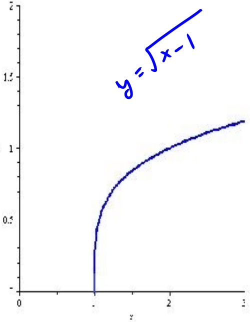
$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1$$

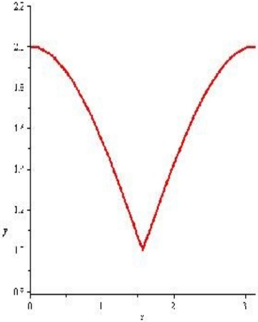
These disagree, so $\lim_{h\to 0} \frac{|h|}{h}$ DNE.

So $f'(0)$ DNE when $f(x) = |x|$.

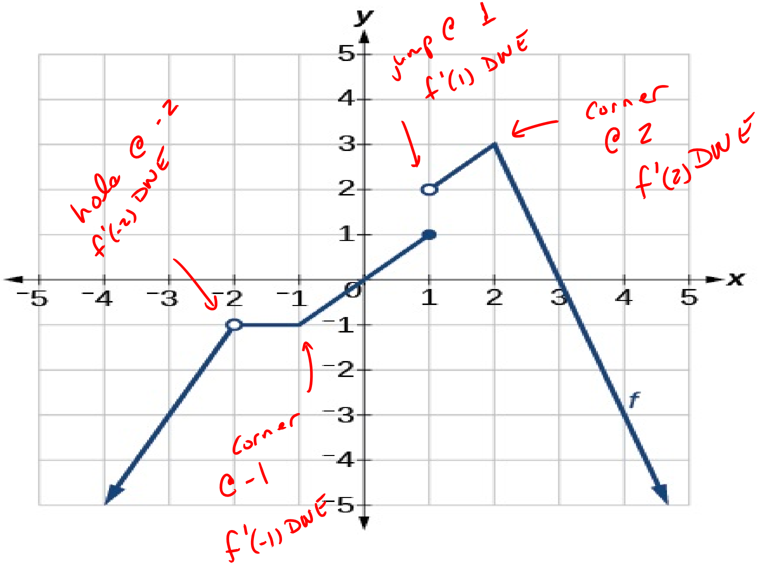
Failure to be differentiable: Discontinuity, Vertical tangent, or Corner/Cusp







Example: **Identify the points were *f* is not differentiable.**



Theorem

Differentiability implies continuity.

That is, if f is differentiable at c, then f is continuous at c. Note that the corner example shows that the converse of this is not true!

for example @ a corner

Questions

(1) **True or False:** Suppose that we know that $f'(3) = 2$. We can conclude that f is continuous at 3.

Differentiability implies continuity.

(2) **True or False:** Suppose that we know that $f'(1)$ does not exist. We can conclude that f is discontinuous at 1.

It may or may not be discontinuous. There could be a corner @ $x=1$.

Section 2.3: The Derivative of a Polynomial; The Derivative of e^x

First some notation:

If $y = f(x)$, the following notation are interchangeable:

$$f'(x) = y'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

$$\text{Leibniz Notation:} \qquad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

You can think of $\quad D, \;$ or $\dfrac{d}{dx}$ $\quad$ as an "operator."

It acts on a function to produce a new function—its derivative.

Taking a derivative is referred to as *differentiation*.

Some Derivative Rules

The derivative of a constant function is zero.

$$\frac{d}{dx}c = 0$$

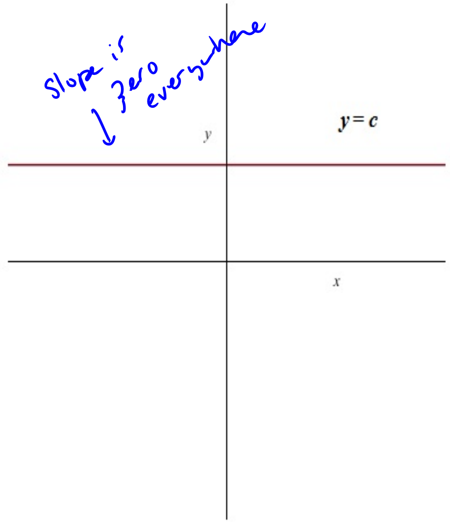
If $f(x) = c$ for constant c, then $f'(x) = 0$.

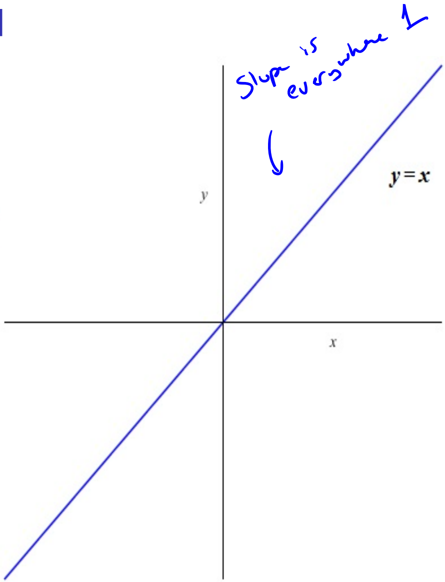
The derivative of the identity function is one.

$$\frac{d}{dx}x = 1$$

If $f(x) = x$ then $f'(x) = \underline{1}$.

$$\frac{d}{dx}c = 0, \qquad \frac{d}{dx}x = 1$$





Evaluate Each Derivative

(a) $\dfrac{d}{dx}(-7) = 0$

Both are constants

(b) $\dfrac{d}{dx}3\pi = 0$

Question

$$\frac{d}{dx}\sqrt{2} =$$

(a) $\sqrt{2}$

(b) 1

(c) 0

$\sqrt{2}$ is a constant.

(d) $\frac{1}{2\sqrt{2}}$

The Power Rule

For positive integer n[1],

$$\frac{d}{dx}x^n = nx^{n-1}$$

This last one is called the **power rule**.

If $f(x) = x^n$ then $f'(x) = nx^{n-1}$

$n = 1, 2, 3, \ldots$

[1]This rule turns out to hold for any real number n, though the proofs for more general cases require results yet to come.

Question

The power rule says that $\frac{d}{dx}x^n = nx^{n-1}$. It follows that

$$\frac{d}{dx}x^6 =$$

(a) nx^5

$n=6$, so $n-1=6-1=5$

(b) $6x^{n-1}$

(c) $6x^5$

(d) $6x$

The power rule (it ain't magic)

Use the binomial expansion

$$(x+h)^6 = x^6 + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h^6$$

to show that $\frac{d}{dx}x^6 = 6x^5$. for $f(x) = x^6$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^6 - x^6}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^6} + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h^6 - \cancel{x^6}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\left(6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5\right)}{\cancel{h}}$$

$$= \lim_{h \to 0} \left(6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5\right)$$

$$= 6x^5 + 0 + 0 + 0 + 0 + 0$$

$$= 6x^5$$

More Derivative Rules

Assume f and g are differentiable functions and k is a constant.

$$\text{Constant multiple rule:} \quad \frac{d}{dx} kf(x) = kf'(x)$$

$$\text{Sum rule:} \quad \frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

$$\text{Difference rule:} \quad \frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

The rules we have thus far allow us to find the derivative of any polynomial function.

Example: Evaluate Each Derivative

(a) $\frac{d}{dx}(x^4 - 3x^2) = \frac{d}{dx}x^4 - \frac{d}{dx}3x^2$

$= \frac{d}{dx}x^4 - 3\frac{d}{dx}x^2$

$= 4x^3 - 3(2x^1) = 4x^3 - 6x$

(b) $\frac{d}{dx}(2x^3+3x^2-12x+1) =$

$$= 2\frac{d}{dx}x^3 + 3\frac{d}{dx}x^2 - 12\frac{d}{dx}x + \frac{d}{dx}1$$

$$= 2(3x^2) + 3(2x^1) - 12\cdot 1 + 0$$

$$= 6x^2 + 6x - 12$$

Example

If $f(x) = 2x^3 + 3x^2 - 12x + 1$, find all points on the graph of f at which the slope of the graph is zero.

From the last example $f'(x) = 6x^2 + 6x - 12$

If the slope of the graph @ $(c, f(c))$ is zero,

then $f'(c) = 0$. Setting $f'(c) = 0$

$6c^2 + 6c - 12 = 0 \Rightarrow 6(c^2 + c - 2) = 0$

$6(c+2)(c-1) = 0 \Rightarrow$ $c = -2$ or $c = 1$

$f(1) = 2\cdot 1^3 + 3\cdot 1^2 - 12\cdot 1 + 1 = -6$

$f(-2) = 2(-2)^3 + 3(-2)^2 - 12\cdot(-2) + 1 = 21$

$\boxed{\text{The two points are } (1,-6) \text{ and } (-2, 21)}$

The Derivative of e^x

Consider $a > 0$ and $a \neq 1$. Let $f(x) = a^x$. Analyze the limit $f'(0)$ and $f'(x)$

Note $f(0) = a^0 = 1$ and $f(0+h) = a^{0+h} = a^h$

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{a^h - 1}{h}$$ provided this limit exists.

For $f(x) = a^x$ then $f(x+h) = a^{x+h} = a^x \cdot a^h$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$

$$= \lim_{h \to 0} a^x \left(\frac{a^h - 1}{h} \right)$$

this is the number $f'(0)$

$$= a^x \left(\lim_{h \to 0} \frac{a^h - 1}{h} \right)$$

$$= f'(0)\, a^x$$

Note this is a constant times a^x

i.e. a constant times $f(x)$

The Derivative of e^x

Definition: The number e is defined[2] by the property

$$\lim_{h\to 0}\frac{e^h-1}{h}=1.$$

It follows that

Theorem: $y=e^x$ is differentiable (at all real numbers) and

$$\frac{d}{dx}e^x=e^x.$$

[2]This is one of several mutually consistent ways to defined this number. Numerically, $e\approx 2.718282$.

Question

Evaluate the derivative of $f(x) = 4x^6 - 2e^x$

power

* x^n has constant power n, the base is variable

(a) $f'(x) = 24x^5 - 2xe^{x-1}$

(b) $f'(x) = 6x^5 - e^x$

* Exponential

e^x variable power x, constant base e

(c) $f'(x) = 24x^5 - 2e^{x-1}$

(d) $f'(x) = 24x^5 - 2e^x$

Section 2.4: Differentiating a Product or Quotient; Higher Order Derivatives

Motivating Example: Evaluate the derivative

$$\frac{d}{dx}[x^3(2x^2-6x+17)]$$

Distribute first

$$= \frac{d}{dx}\left(2x^5 - 6x^4 + 17x^3\right)$$

$$= 10x^4 - 24x^3 + 51x^2$$

Derivative of A Product

Now consider evaluating the derivative

$$\frac{d}{dx}[(3x^5-2x^2+x)(x^3-2x^2+x-1)]$$

We can take the same approach here,

but the algebra is more tedious.

Derivative of A Product

Theorem: (Product Rule) Let f and g be differentiable functions of x. Then the product $f(x)g(x)$ is differentiable. Moreover

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

This can be stated using Leibniz notation as

$$\frac{d}{dx}[f(x)g(x)] = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}.$$

Example

Compute $\frac{d}{dx}x^5$ using the product rule with $f(x) = x^2$ and $g(x) = x^3$. Compare this with the result from the power rule on x^5.

By the power rule $\frac{d}{dx}x^5 = 5x^4$.

Using the product rule

$$\frac{d}{dx}x^5 = \frac{d}{dx}\left[x^2 \cdot x^3\right] = \left(\frac{d}{dx}x^2\right)x^3 + x^2\left(\frac{d}{dx}x^3\right)$$

$$= (2x^1)\, x^3 + x^2(3x^2)$$

$$= 2x^4 + 3x^4 = 5x^4$$

Example

$$\frac{d}{dx}[f(x)\,g(x)] = f'(x)\,g(x) + f(x)\,g'(x)$$

Evaluate $\dfrac{d}{dx}[(3x^5-2x^2+x)(x^3-2x^2+x-1)]$

Let $f(x) = 3x^5-2x^2+x$, $f'(x) = 15x^4-4x+1$

$g(x) = x^3-2x^2+x-1$, $g'(x) = 3x^2-4x+1$

$$\frac{d}{dx}\left[(3x^5-2x^2+x)(x^3-2x^2+x-1)\right]$$

$$= (15x^4-4x+1)(x^3-2x^2+x-1) + (3x^5-2x^2+x)(3x^2-4x+1)$$

Example

Evaluate $\frac{d}{dx} e^{2x}$ using the product rule.

Note $e^{2x} = e^{x+x} = e^{x} \cdot e^{x}$

So $\frac{d}{dx}\left[e^{2x}\right] = \frac{d}{dx}\left[e^{x} \cdot e^{x}\right]$

$= \left(\frac{d}{dx} e^{x}\right) e^{x} + e^{x} \left(\frac{d}{dx} e^{x}\right)$

$= e^{x} \cdot e^{x} + e^{x} \cdot e^{x} = e^{2x} + e^{2x} = 2e^{2x}$

Question

$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$; $\frac{d}{dx}[e^{2x}] = 2e^{2x}$

Evaluate $f'(x)$ where $f(x) = 3x^4e^{2x}$.

(a) $f'(x) = 6x^4e^{2x}$

(b) $f'(x) = 12x^3e^{2x} + 6x^4e^{2x}$

(c) $f'(x) = 24x^3e^{2x}$

(d) $f'(x) = 3x^4e^{2x} + 12x^3e^{2x}$

The Derivative of a Quotient

Theorem (Quotient Rule) Let f and g be differentiable functions of x. Then on any interval for which $g(x) \neq 0$, the ratio $\frac{f(x)}{g(x)}$ is differentiable. Moreover

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

This can be stated using Leibniz notation as

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx}g(x) - f(x)\frac{dg}{dx}}{[g(x)]^2}.$$

A Special Case

An immediate consequence of this is that

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{[g(x)]^2}.$$

Evaluate $\quad \dfrac{d}{dx}x^{-1} = \dfrac{d}{dx}\dfrac{1}{x}$

here $g(x) = x$, so $g'(x) = 1$

$$= -\frac{1}{(x)^2} = \frac{-1}{x^2} = -x^{-2}$$

This is the power rule w/ $n = -1$

Example

Use the quotient rule to show that for positive integer n[3]

$$\frac{d}{dx}x^{-n} = -nx^{-n-1}$$

$$\frac{d}{dx}x^{-n} = \frac{d}{dx}\frac{1}{x^n}$$

for $g(x) = x^n$, $g'(x) = nx^{n-1}$

and $(g(x))^2 = (x^n)^2 = x^{2n}$

$$= -\frac{nx^{n-1}}{(x^n)^2}$$

$$= -\frac{nx^{n-1}}{x^{2n}}$$

[3]Note that this shows that the power rule works for both positive and negative integers.

$$= -n\, x^{n-1-2n}$$

$$= -n\, x^{-n-1}$$

That is, $\dfrac{d}{dx} x^{-n} = -n\, x^{-n-1}$

For example $\dfrac{d}{dx} x^{-4} = -4x^{-5}$

Question

$$\frac{d}{dx}\frac{1}{g(x)} = -\frac{g'(x)}{(g(x))^2}$$

Evaluate $\quad \dfrac{d}{dx}e^{-x} = \dfrac{d}{dx}\,\dfrac{1}{e^x}$

$$\frac{d}{dx}\frac{1}{e^x} = -\frac{e^x}{(e^x)^2} = -\frac{e^x}{e^{2x}}$$

$$= -e^{x-2x} = -e^{-x}$$

(a) $-e^{-x}$

(b) e^{-x}

(c) $\frac{1}{e^x}$

(d) can't be determined without more information