February 15 MATH 1112 sec. 54 Spring 2019

Section 5.5: Solving Exponential and Logarithmic Equations

Base-Exponent Equality For any $a > 0$ with $a \neq 1$, and for any real numbers x and y

$$a^x = a^y \quad \text{if and only if} \quad x = y.$$

Logarithm Equality For and $a > 0$ with $a \neq 1$, and for any positive numbers x and y

$$\log_a x = \log_a y \quad \text{if and only if} \quad x = y.$$

Inverse Function For any $a > 0$ with $a \neq 1$

$$a^{\log_a x} = x \quad \text{for every} \quad x > 0$$

$$\log_a(a^x) = x \quad \text{for every real} \quad x.$$

Logarithm Equations

We can also use the three log properties to solve equations involving logarithms.

For any $a > 0$ with $a \neq 1$, and M and N positive numbers and r any real number

$$\log_a(MN) = \log_a M + \log_a N$$

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$$

$$\log_a(M^r) = r\log_a M$$

Log Equations & Verifying Answers

Double checking answers is always recommended. When dealing with functions whose domains are restricted, **answer verification** is critical.

Use properties of logarithms to solve the equation

$$\ln x + \ln(x-3) = \ln 2 + \ln(4x-15)$$

using log properties

$$\ln\left(x(x-3)\right) = \ln\left(2(4x-15)\right)$$

By log equivalence

$$x(x-3) = 2(4x-15)$$

Just algebra

$$x^2 - 3x = 8x - 30$$

$$x^2 - 3x - 8x + 30 = 0$$

$$x^2 - 11x + 30 = 0$$

$$(x-6)(x-5) = 0$$

The quadratic equation has 2 solutions

$x = 6$ or $x = 5$.

We need to verify that these solve the original equation

$$\ln x + \ln(x-3) = \ln 2 + \ln(4x-15)$$

Check $x = 5$

$$\ln 5 + \ln(5-3) \stackrel{?}{=} \ln 2 + \ln(4\cdot5-15)$$

$$\ln 5 + \ln 2 \stackrel{?}{=} \ln 2 + \ln 5$$

Check $x=6$

$$\ln 6 + \ln(6-3) \stackrel{?}{=} \ln 2 + \ln(4\cdot 6 - 15)$$

$$\ln 6 + \ln 3 \stackrel{?}{=} \ln 2 + \ln 9$$

$$\ln(6\cdot 3) \stackrel{?}{=} \ln(2\cdot 9)$$

yes $\ln 18 = \ln 18$

Both do solve the equation. There are two solutions 5 and 6.

Question

Solve the equation $\log_6 x + \log_6(x-1) = 1$. (Hint: $\log_6 6 = 1$.)

(a) $x = 3$ or $x = -2$

(b) $x = 2$ or $x = -3$

(c) $x = 3$

(d) $x = 2$

(e) $x = 0$ or $x = 1$

Combining Skills

Find all solutions of the equation[1]

$$\frac{e^x + e^{-x}}{2} = 2$$

We can multiply by 2

$$e^x + e^{-x} = 4$$

Note $\ln(e^x + e^{-x})$ doesn't simplify! So we

won't take the log here.

Instead, we'll multiply by e^x

[1] The function $f(x) = \frac{e^x+e^{-x}}{2}$ is a well known function called the *hyperbolic cosine*.

$$e^x\left(e^x + e^{-x}\right) = e^x(4)$$

$$\left(e^x\right)^2 + e^x e^{-x} = 4e^x$$

$* \quad e^{-x} = \frac{1}{e^x}$ so $e^x e^{-x} = e^x \cdot \frac{1}{e^x} = \frac{e^x}{e^x} = 1$

$$\left(e^x\right)^2 + 1 = 4e^x \implies \left(e^x\right)^2 - 4e^x + 1 = 0$$

Let $u = e^x$. The equation is

$u^2 - 4u + 1 = 0$ a quadratic equation.

By the quadratic formula

$$u = \frac{4 \pm \sqrt{(-4)^2 - 4\cdot(1)(1)}}{2} = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

we have 2 u's

$$u = 2 + \sqrt{3} \quad \text{or} \quad u = 2 - \sqrt{3}$$

$$u = e^x \quad \text{so} \quad x = \ln u$$

Both u's are positive, so there are

2 solutions

$$x = \ln(2+\sqrt{3}) \quad \text{or} \quad x = \ln(2-\sqrt{3}).$$