February 15 Math 2306 sec. 54 Spring 2019

Section 6: Linear Equations Theory and Terminology

We defined **linear dependence** and the Wronskian last time. The following theorem provides a test for linear dependence (or independence).

Theorem (a test for linear independence) Let $f_1, f_2, \dots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \dots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

Determine if the functions are linearly dependent or independent:

$$y_1 = x^2, \quad y_2 = x^3 \quad I = (0, \infty)$$

We computed the Wronskian and found

$$W(y_1, y_2)(x) = x^4$$

which is not the zero function. So the functions are linearly **independent**.

Fundamental Solution Set

We're still considering this equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

with the assumptions $a_n(x)\neq 0$ and $a_i(x)$ are continuous on I.

Definition: A set of functions $y_1, y_2, \dots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,
(ii) there are n of them, and
(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of n^{th} order Linear Homogeneous Equation

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = x^2$ and $y_2 = x^3$ form a fundamental solution set of the ODE

$$x^2 y'' - 4xy' + 6y = 0 \quad \text{on} \quad (0, \infty),$$

and determine the general solution.

We have to show that there are

- 2 solutions,
- they are actual solutions
- they are linearly independent

There are 2 of them for this 2^{nd} order equation, and they are linearly independent since $W(y_1, y_2)(x) = x^4 \neq 0$.

We have to show that they solve the ODE.

$$x^2y'' - 4xy' + 6y = 0$$

Verify y_1:

$y_1 = x^2$

$y_1' = 2x$

$y_1'' = 2$

$$x^2y_1'' - 4xy_1' + 6y_1 =$$

$$x^2(2) - 4x(2x) + 6x^2 =$$

$$2x^2 - 8x^2 + 6x^2 = 0$$

$$0 = 0$$

y_1 is a solution

Verify y_2:

$y_2 = x^3$

$y_2' = 3x^2$

$y_2'' = 6x$

$$x^2y_2'' - 4xy_2' + 6y_2 =$$

$$x^2(6x) - 4x(3x^2) + 6x^3 =$$

$$6x^3 - 12x^3 + 6x^3 = 0$$

$$0 = 0$$

y_2 is also a solution

Hence y_1, y_2 is a fundamental solution set.

The general solution has the form

$$y = c_1 y_1 + c_2 y_2$$

The general solution is

$$y = c_1 x^2 + c_2 x^3$$

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

This has identical left side and zero right side.

Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$! (complementary plus particular) The complemenatry part

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

Another Superposition Principle (for nonhomogeneous eqns.)

Let y_{p_1}, y_{p_2}, ..., y_{p_k} be k particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g_i(x)$$

for $i=1,\dots,k$. Assume the domain of definition for all k equations is a common interval I.

Then

$$y_p=y_{p_1}+y_{p_2}+\cdots+y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)\frac{d^ny}{dx^n}+\cdots+a_0(x)y=g_1(x)+g_2(x)+\cdots+g_k(x).$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

We will construct the general solution by considering sub-problems.

(a) **Part 1** Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

Substitute

$y_{p_1} = 6$

$y_{p_1}' = 0$

$y_{p_1}'' = 0$

$$x^2 y_{p_1}'' - 4x y_{p_1}' + 6 y_{p_1} \stackrel{?}{=} 36$$

$$x^2(0) - 4x(0) + 6(6) \stackrel{?}{=} 36$$

$$36 = 36$$

Yes y_{p_1} solves the equation

$$x^2y'' - 4xy' + 6y = 36$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

$g_1(x) = 36$ $g_2(x) = -14x$

(b) **Part 2** Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

Substitute

$y_{p_2} = -7x$

$y_{p_2}' = -7$

$y_{p_2}'' = 0$

$$x^2 y_{p_2}'' - 4x y_{p_2}' + 6y_{p_2} \stackrel{?}{=} -14x$$

$$x^2(0) - 4x(-7) + 6(-7x) \stackrel{?}{=} -14x$$

$$28x - 42x \stackrel{?}{=} -14x$$

$$-14x = -14x$$

y_{p_2} does solve $x^2y'' - 4xy' + 6y = -14x$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) **Part 3** We already know that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

$y_{p_1} = 6 \qquad y_{p_2} = -7x$

By superposition, $y_p = y_{p_1} + y_{p_2} = 6 - 7x$

Also, $y_c = c_1 y_1 + c_2 y_2 = c_1 x^2 + c_2 x^3$

The general solution is $y_c + y_p$,

$$y = c_1 x^2 + c_2 x^3 + 6 - 7x$$

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

we have the general solution to the ODE

$$y = c_1x^2 + c_2x^3 + 6 - 7x$$

we must apply the initial conditions.

$$y' = 2c_1x + 3c_2x^2 - 7$$

$$y(1) = c_1 1^2 + c_2 1^3 + 6 - 7(1) = 0$$

$$c_1 + c_2 - 1 = 0$$

$$c_1 + c_2 = 1$$

$$y'(1) = 2c_1(1) + 3c_2 1^2 - 7 = -5$$

$$2c_1 + 3c_2 - 7 = -5$$

$$2c_1 + 3c_2 = 2$$

We solve the system $\left.\begin{array}{l} c_1 + c_2 = 1 \\ 2c_1 + 3c_2 = 2 \end{array}\right\} \Rightarrow$ $c_1 = 1$ and $c_2 = 0$

The solution to the IVP is

$$\boxed{y = x^2 + 6 - 7x}$$