Section 7: Reduction of Order

We’ll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

\[ a_2(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)y = 0. \]

Let us assume that \( a_2(x) \neq 0 \) on the interval of interest. We will write our equation in standard form

\[ \frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0 \]

where \( P = \frac{a_1}{a_2} \) and \( Q = \frac{a_0}{a_2} \).
\[
\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0
\]

Recall that every fundamental solution set will consist of two linearly independent solutions \(y_1\) and \(y_2\), and the general solution will have the form

\[
y = c_1y_1(x) + c_2y_2(x).
\]

Suppose we happen to know one solution \(y_1(x)\). **Reduction of order** is a method for finding a second linearly independent solution \(y_2(x)\) that starts with the assumption that

\[
y_2(x) = u(x)y_1(x)
\]

for some function \(u(x)\). The method involves finding the function \(u\).

Since \(y_1, y_2\) are to be linearly independent, \(u(x)\) cannot be a constant function.
Example

Verify that $y_1 = e^{-x}$ is a solution of $y'' - y = 0$. Then find a second solution $y_2$ of the form

$$y_2(x) = u(x)y_1(x) = e^{-x}u(x).$$

Confirm that the pair $y_1, y_2$ is linearly independent.

Let's substitute $y_2$ into the ODE.

$$y_2 = e^{-x}u$$
$$y_2' = e^{-x}u' + (-e^{-x})u = e^{-x}u' - e^{-x}u$$
$$y_2'' = e^{-x}u'' - e^{-x}u' - e^{-x}u' + e^{-x}u$$
$$= e^{-x}u'' - 2e^{-x}u' + e^{-x}u$$

$$y_2'' - y_2 = 0$$
\[ e^{-x} u'' - 2e^{-x} u' + e^{-x} u - e^{-x} u = 0 \]

\[ e^{-x} u'' - 2e^{-x} u' = 0 \]

\[ e^{-x} (u'' - 2u') = 0 \Rightarrow u'' - 2u' = 0 \]

Let \( w = u' \) then \( w' = u'' \), so \( w \) solves a 1st order ODE

\[ w' - 2w = 0 \]

This is separable

\[ \frac{dw}{dx} = 2w \Rightarrow \frac{1}{w} \frac{dw}{dx} = 2 \]

assuming \( w \neq 0 \)
\[ \int \frac{1}{w} \, dw = \int 2 \, dx \Rightarrow \ln|w| = 2x + C \]

**Exponentiate**
\[ |w| = e^{2x+C} = e^C e^{2x} \]

**Letting** \( k = e^C \) or \( -e^C \)
\[ w = k e^{2x} \]

\( w = u' \) so \( u = \int w \, dx = \int k e^{2x} \, dx \)
\[ u = \frac{k}{2} \, e^{2x} + C_1 \]

\( y_2 = u y_1 \).
Now, the general solution will be

\[ y = c_1 y_1 + c_2 y_2. \]

Taking \( u_2 \):

\[ \left( \frac{k}{2} e^{2x} + C_1 \right) e^{-x} = \frac{k}{2} e^{2x} e^{-x} + C_1 e^{-x} \]

we can take \( \frac{k}{2} = 1 \) because we'll include the coefficient \( C_2 \). And, we can ignore the \( e^{-x} \) term since this will also be included in the general solution.

So we can take \( u(x) = e^{2x} \)
\[ y_2 = uy_1 = e^{2x}e^{-x} = e^x \]

The fundamental solution set is
\[ y_1 = e^{-x} \quad \text{and} \quad y_2 = e^x \]

The general solution to \( y'' - y = 0 \) is
\[ y = c_1 e^{-x} + c_2 e^x \]
Generalization

Consider the equation in standard form with one known solution. Determine a second linearly independent solution.

\[
d\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - - - \text{is known.}
\]

Note \(y_1\) being a solution means \(y_1'' + P(x)y_1' + Q(x)y_1 = 0\).

Assume a second solution

\[
y_2 = u(x)y_1, \quad \text{for some } u(x)
\]

\[
y_2' = u'y_1 + uy_1',
\]

\[
y_2'' = u''y_1 + 2u'y_1' + uy_1''
\]

\[
= u''y_1 + 2u'y_1' + uy_1''
\]
\[ y_2'' + P(x) y_2' + Q(x) y_2 = 0 \]

\[ u'' y_1 + 2 u' y_1' + u y_1'' + P(x) (u' y_1 + u y_1') + Q(x) u y_1 = 0 \]

Collect by \( u, u', u'' \)

\[ u'' y_1 + (2y_1' + P(x) y_1) u' + (y_1'' + P(x) y_1' + Q(x) y_1) u = 0 \]

\[ y_1 u'' + (2y_1' + P(x) y_1) u' = 0 \]

Let \( w = u' \), then \( w' = u'' \) and \( w \) solves the 1st order ODE.
\[ y, w' + (2y, + p(x)y, ) w = 0 \]

We'll assume it's safe to divide by \( y, \) and separate variables.

\[ y, \frac{dw}{dx} = -(2y, + p(x)y, ) w \]

\[ \frac{1}{w} \frac{dw}{dx} = -\frac{(2y, + p(x)y, )}{y,} = -2 \frac{dy,}{y,} - p(x) \]

\[ \int \frac{1}{w} dw = \int -2 \frac{dy,}{y,} dx - \int p(x) dx \]

\[ \int \frac{1}{w} dw = \int -2 \frac{dy,}{y,} - \int p(x) dx \]
\[
\ln |w| = \ln |y_1| - \mathcal{P}(x) \, dx + C
\]

\[
\exp \text{ to}
\]

\[
|w| = e^{\ln |y_1| - \mathcal{P}(x) \, dx + C} = e^{\ln y_1} e^{-\mathcal{P}(x) \, dx}
\]

Absorbing any sign into the constant \( e^c \)

\[
W = \frac{e}{(y_1)^2}
\]
\[ w = w' \quad \text{so integrate again to get} \]

\[ u = \int -\frac{e}{(y_1(x))^2} \, dx \]

Finally,

\[ y_2 = uy_1 = y_1(x) \int -\frac{e}{(y_1(x))^2} \, dx \]
Reduction of Order Formula

For the second order, homogeneous equation in standard form with one known solution $y_1$, a second linearly independent solution $y_2$ is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P(x) \, dx}}{(y_1(x))^2} \, dx$$
Example

Find the general solution of the ODE given one known solution

\[ x^2 y'' - 3xy' + 4y = 0, \quad y_1 = x^2 \]

Standard form:

\[ y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0 \]

\[ P(x) = \frac{-3}{x} \quad \text{so} \quad e^\left( -\int P(x) \, dx \right) = e^{\int \frac{-3}{x} \, dx} = e^{\int \frac{3}{x} \, dx} = e^{3\ln x} = e^{\ln x^3} = x^3 \]
\[ u = \int \frac{e^{- \int \text{frac{dy}{y_1}}} dx = \int \frac{x^3}{(x^2)^2} dx \]

\[ = \int \frac{x^3}{x^4} dx = \int \frac{1}{x} dx = \ln x \quad \text{Supposing} \quad x > 0 \]

\[ y_2 = y_1, u = x^2 \ln x \]

The general solution to the ODE is

\[ y = c_1 y_1 + c_2 y_2 \Rightarrow y = c_1 x^2 + c_2 x^2 \ln x \]