

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)y = 0.$$

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0$$

where $P = a_1/a_2$ and $Q = a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Recall that every fundamental solution set will consist of two linearly independent solutions y_1 and y_2 , and the general solution will have the form

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

y_1 is known,

$$y_2(x) = u(x)y_1(x) \quad \text{Task: find } u(x)$$

for some function $u(x)$. The method involves finding the function u .

Since y_1, y_2 are to be linearly independent,

$u(x)$ cannot be a constant function.

Example

Verify that $y_1 = e^{-x}$ is a solution of $y'' - y = 0$. Then find a second solution y_2 of the form

$$y_2(x) = u(x)y_1(x) = e^{-x}u(x).$$

Confirm that the pair y_1, y_2 is linearly independent.

Let's substitute y_2 into the ODE.

$$y_2 = e^{-x}u$$

$$y_2' = e^{-x}u' + (-e^{-x})u = e^{-x}u' - e^{-x}u$$

$$y_2'' = e^{-x}u'' - e^{-x}u' - e^{-x}u' + e^{-x}u$$

$$= e^{-x}u'' - 2e^{-x}u' + e^{-x}u$$

$$y_2'' - y_2 = 0$$

$$e^x u'' - 2e^x u' + e^x u - e^x u = 0$$

$$e^x u'' - 2e^x u' = 0$$

$$e^x(u'' - 2u') = 0 \Rightarrow u'' - 2u' = 0$$

Let $w = u'$ then $w' = u''$, so w solves a
1st order ODE

$$w' - 2w = 0$$

This is separable

assuming
 $w \neq 0$

$$\frac{dw}{dx} = 2w \Rightarrow \frac{1}{w} \frac{dw}{dx} = 2$$

$$\int \frac{1}{w} dw = \int 2dx \Rightarrow \ln|w| = 2x + C$$

Exponentiate

$$|w| = e^{2x+C} = e^C e^{2x}$$

Letting $k = e^C$ or $-e^C$

$$w = k e^{2x}$$

$$w = u \quad \text{so} \quad u = \int w dx = \int k e^{2x} dx$$

$$u = \frac{k}{2} e^{2x} + C_1$$

$$y_2 = u y_1$$

* Note, the general solution will be

$$y = C_1 y_1 + C_2 y_2 .$$

Taking u_1 : $(\frac{k}{2} e^{zx} + C_1) e^{-x} = \frac{k}{2} e^{zx-x} e + C_1 e^{-x}$

We can take $\frac{k}{2} = 1$ because we'll include the coefficient C_2 . And, we can ignore the e^{-x} term since this will also be included in the general solution.

So we can take $u(x) = e^{zx}$

$$y_2 = u y_1 = e^{-x} \cdot e^{2x} = e^x$$

The fundamental solution set is

$$y_1 = e^{-x} \quad \text{and} \quad y_2 = e^x$$

The general solution to $y'' - y = 0$ is

$$y = C_1 e^{-x} + C_2 e^x$$

Generalization

Consider the equation **in standard form** with one known solution.
Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) \text{ --- is known.}$$

Note y_1 being a solution means $y_1'' + P(x)y_1' + Q(x)y_1 = 0$

Assume a second solution

$$y_2 = u y_1, \quad \text{for some } u(x)$$

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$$

$$= u''y_1 + 2u'y_1' + uy_1''$$

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

$$u''y_1 + 2u'y_1' + uy_1'' + P(x)(u'y_1 + uy_1') + Q(x)uy_1 = 0$$

Collect by u, u', u''

$$u''y_1 + (2y_1' + P(x)y_1)u' + \underbrace{(y_1'' + P(x)y_1' + Q(x)y_1)}_{0''}u = 0$$

y_1 is a solution

$$y_1u'' + (2y_1' + P(x)y_1)u' = 0$$

Let $w = u'$, then $w' = u''$ and w solves the
1st order ODE

$$y_1 w' + (zy_1' + p(x)y_1)w = 0$$

Well assume it's safe to divide by y_1 and
separate variables

$$y_1 \frac{dw}{dx} = - (zy_1' + p(x)y_1)w$$

$$\frac{1}{w} \frac{dw}{dx} = - \frac{(zy_1' + p(x)y_1)}{y_1} = -2 \frac{\frac{dy_1}{dx}}{y_1} - p(x)$$

$$\int \frac{1}{w} dw = \int -2 \frac{\frac{dy_1}{dx}}{y_1} dx - \int p(x) dx$$

$$\int \frac{1}{w} dw = \int -2 \frac{dy_1}{y_1} - \int p(x) dx$$

$$\ln |w| = -2 \ln |y_1| - \int p(x) dx + C$$

exponentiate

$$-2 \ln |y_1| - \int p(x) dx + C$$

$$|w| = e$$

$$= e^C y_1^{-2} e^{-\int p(x) dx}$$

Absorbing any sign into the constant e^C

$$- \int p(x) dx$$

$$w = \frac{e}{(y_1)^2}$$

$w = u'$ so integrate again to

get

$$u = \int \frac{e^{-\int p(x)dx}}{(y_1(x))^2} dx$$

Finally,

$$y_2 = uy_1 = y_1(x)$$

$$\int \frac{e^{-\int f(x)dx}}{(y_1(x))^2} dx$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1 , a second linearly independent solution y_2 is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P(x) dx}}{(y_1(x))^2} dx$$

Example

Find the general solution of the ODE given one known solution

$$x^2y'' - 3xy' + 4y = 0, \quad y_1 = x^2$$

Standard form:

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

$$P(x) = \frac{-3}{x} \quad \text{so} \quad e^{-\int P(x)dx} = e^{-\int \frac{-3}{x} dx}$$

$$= e^{\int \frac{3}{x} dx} = e^{3\ln x} = e^{\ln x^3}$$

$$= x^3$$

$$u = \int \frac{e^{-\int p(x)dx}}{(y_1)^2} dx = \int \frac{x^3}{(x^2)^2} dx$$

$$= \int \frac{x^3}{x^4} dx = \int \frac{1}{x} dx = \ln x$$

Supposing
 $x > 0$

$$y_2 = y_1, u = x^2 \ln x$$

The general solution to the ODE is

$$y = C_1 y_1 + C_2 y_2 \Rightarrow y = C_1 x^2 + C_2 x^2 \ln x$$