February 15 Math 2306 sec. 60 Spring 2018

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Let us assume that $a_2(x)\neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

where $P=a_1/a_2$ and $Q=a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1y_1(x) + c_2y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

y_1 - known

u - want to find

$$y_2(x) = u(x)y_1(x)$$

for some function $u(x)$. The method involves finding the function u.

Due to linear independence, $u(x)$ cannot be a constant function.

Example

Verify that $y_1 = e^{-x}$ is a solution of $y'' - y = 0$. Then find a second solution y_2 of the form

left as an exercise

$$y_2(x) = u(x)y_1(x) = e^{-x}u(x).$$

Confirm that the pair y_1, y_2 is linearly independent.

y_2 is a solution, so let's substitute it into $y_2'' - y_2 = 0$

$$y_2 = e^{-x} u$$

$$y_2' = e^{-x} u' - e^{-x} u$$

$$y_2'' = e^{-x} u'' - e^{-x} u' - e^{-x} u' + e^{-x} u$$

$$= e^{-x} u'' - 2e^{-x} u' + e^{-x} u$$

$$y_2'' - y_2 = 0$$

$$e^{-x}u'' - 2e^{-x}u' + \cancel{e^{-x}u} - \cancel{e^{-x}u} = 0$$

$$e^{-x}u'' - 2e^{-x}u' = 0$$

$$e^{-x}(u'' - 2u') = 0 \Rightarrow u'' - 2u' = 0$$

Let $w = u'$, then $w' = u''$. w satisfies

$$w' - 2w = 0 \qquad \text{1st order.}$$

$$\frac{dw}{dx} = 2w$$

$$\frac{1}{w}\frac{dw}{dx} = 2 \implies \int \frac{1}{w}\,dw = \int 2\,dx$$

$$\ln|w| = 2x + k \implies |w| = e^{2x+k} = e^{k}e^{2x}$$

Letting $C = e^{k}$ or $-e^{k}$

$$w = Ce^{2x}$$

$u' = w$ so $\quad u = \int w\,dx = \int Ce^{2x}\,dx$

$$= \frac{1}{2}Ce^{2x} + B$$

$$y_2 = uy_1$$

* The general solution is $y = c_1 y_1 + c_2 y_2$

From $\left(\frac{1}{2}Ce^{2x} + B\right)e^{-x} = \frac{1}{2}Ce^{2x}e^{-x} + Be^{-x}$

we're going to include $c_1 e^{-x}$, so we can take $B = 0$. We'll also multiply y_2 by c_2, so we can take $\frac{1}{2}C$ to be 1.

We'll take $u = e^{2x}$.

Then $y_2 = uy_1 = e^{2x} \cdot e^{-x} = e^{x}$

The general solution to $y'' - y = 0$ is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 e^{-x} + c_2 e^{x}$$

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - -\text{is known.}$$

y_1 is a solution, hence $\quad y_1'' + P(x)y_1' + Q(x)y_1 = 0$

$y_2 = uy_1$

$y_2' = u'y_1 + uy_1'$

$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$

$= u''y_1 + 2u'y_1' + uy_1''$

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

$$u''y_1 + 2u'y_1' + uy_1'' + P(x)\left(u'y_1 + uy_1'\right) + Q(x)uy_1 = 0$$

Collect u, u', u'' terms

$$u''y_1 + \left(2y_1' + P(x)y_1\right)u' + \big(\underbrace{y_1'' + P(x)y_1' + Q(x)y_1}_{0 \text{ since } y_1 \text{ is a solution}}\big)u = 0$$

so $$y_1 u'' + \left(2y_1' + P(x)y_1\right)u' = 0$$

Let $w = u'$, so $w' = u''$ the w solves

$$y_1 w' + (2y_1' + P(x)y_1)w = 0$$

Separate variables

$$y_1 \frac{dw}{dx} = -(2y_1' + P(x)y_1)w$$

$$\frac{1}{w}\frac{dw}{dx} = \frac{-(2y_1' + P(x)y_1)}{y_1}$$

$$\frac{1}{w}\frac{dw}{dx} = -2\frac{\frac{dy_1}{dx}}{y_1} - P(x)$$

$$\frac{1}{w}\frac{dw}{dx}dx = -2\frac{\frac{dy_1}{dx}}{y_1}dx - P(x)\,dx$$

$$\int \frac{1}{w}\,dw = \int -2\,\frac{dy_1}{y_1} - \int p(x)\,dx$$

$$\ln w = -2\ln|y_1| - \int p(x)\,dx$$

assuming $w>0$

$$w = e^{-2\ln|y_1| - \int p(x)\,dx}$$

$$= y_1^{-2}\cdot e^{-\int p(x)\,dx} = \frac{e^{-\int p(x)\,dx}}{(y_1)^2}$$

$$w = u' \Rightarrow u = \int w\,dx$$

$$u = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Then $y_2 = u y_1$

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the general solution of the ODE given one known solution

$$x^2y'' - 3xy' + 4y = 0, \quad y_1 = x^2$$

Assume $x > 0$. Standard form

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

$$P(x) = \frac{-3}{x}$$

$$e^{-\int P(x)dx} = e^{-\int \frac{-3}{x}dx} = e^{\int \frac{3}{x}dx}$$

$$= e^{3\ln x} = e^{\ln x^3} = x^3$$

$$u = \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx = \int \frac{x^3}{(x^2)^2}\,dx$$

$$= \int \frac{x^3}{x^4}\,dx = \int \frac{1}{x}\,dx = \ln x$$

$$y_2 = u y_1 = x^2 \ln x$$

The general solution to the ODE is

$$y = C_1 y_1 + C_2 y_2$$

$$y = c_1 x^2 + c_2 x^2 \ln x$$