February 15 Math 2306 sec. 60 Spring 2019

Section 6: Linear Equations Theory and Terminology

We defined **linear dependence** and the Wronskian last time. The following theorem provides a test for linear dependence (or independence).

Theorem (a test for linear independence) Let $f_1, f_2, ..., f_n$ be n - 1 times continuously differentiable on an interval *I*. If there exists x_0 in *I* such that $W(f_1, f_2, ..., f_n)(x_0) \neq 0$, then the functions are **linearly independent** on *I*.

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Determine if the functions are linearly dependent or independent:

$$y_1 = x^2$$
, $y_2 = x^3$ $I = (0, \infty)$

We computed the Wronskian and found

 $W(y_1,y_2)(x)=x^4$

which is not the zero function. So the functions are linearly **independent**.

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Fundamental Solution Set

We're still considering this equation

$$a_n(x)rac{d^n y}{dx^n} + a_{n-1}(x)rac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)rac{dy}{dx} + a_0(x)y = 0$$

with the assumptions $a_n(x) \neq 0$ and $a_i(x)$ are continuous on *I*.

Definition: A set of functions $y_1, y_2, ..., y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

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- (i) are solutions of the equation,
- (ii) there are *n* of them, and
- (iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of *n*th order Linear Homogeneous Equation

Let $y_1, y_2, ..., y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

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where c_1, c_2, \ldots, c_n are arbitrary constants.

Example

Verify that $y_1 = x^2$ and $y_2 = x^3$ form a fundamental solution set of the ODE

$$x^2y'' - 4xy' + 6y = 0$$
 on $(0, \infty)$,

and determine the general solution.

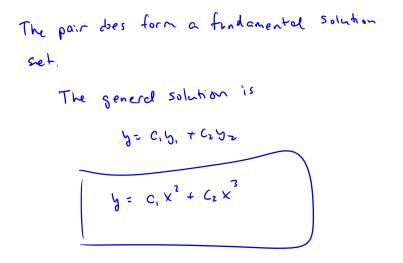
We want to show that
. there are 2 functions in our set
. they are solutions
. they are linearly independent
There are two of them, y, and yz. We
found that
$$W(y_1, y_2)(x) = x^4 \pm 0$$
 so
they are linearly in dependent.
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Checke
$$y_{1} = x^{2}$$
 $x^{2}y_{1}'' - 4xy_{1}' + 6y_{1} =$
 $y_{1}' = 2x$ $x^{2}(z_{1}) - 4x(z_{2}x) + 6x^{2} =$
 $y_{1}'' = 2$ $2x^{2} - 8x^{2} + 6x^{2} = 0$ y_{1} is a solution

Check
$$y_2: y_2 = x^3$$
 $x^2y_2'' - 4xy_2' + 6y_2 =$
 $y_2' = 3x^2$ $x^2(6x) - 4x(3x^2) + 6x^3 =$ y_2 is also
 $y_2'' = 6x$ $6x^3 - 12x^3 + 6x^3 = 0$ a solution

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Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where *g* is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and *g* are continuous.

The associated homogeneous equation is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

The left side is exactly the same as the

non honogeneous version.

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Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1 , y_2, \ldots, y_n be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where c_1, c_2, \ldots, c_n are arbitrary constants.

Note the form of the solution $y_c + y_p!$ (complementary plus particular) The complemenatry part

 $V_{c}(x) = c_{1} V_{1}(x) + c_{2} V_{2}(x) + \cdots + c_{n} V_{n}(x)$

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Another Superposition Principle (for nonhomogeneous eqns.)

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

for i = 1, ..., k. Assume the domain of definition for all k equations is a common interval I.

Then

$$y_{\rho}=y_{\rho_1}+y_{\rho_2}+\cdots+y_{\rho_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)rac{d^ny}{dx^n}+\cdots+a_0(x)y=g_1(x)+g_2(x)+\cdots+g_k(x).$$

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Example $x^2y'' - 4xy' + 6y = 36 - 14x$ We will construct the general solution by considering sub-problems. (a) Part 1 Verify that

$$y_{p_{1}} = 6 \quad \text{solves} \quad x^{2}y'' - 4xy' + 6y = 36.$$
Substitute
$$x^{2}y_{p_{1}'} - 4xy' + 6y_{p_{1}} = 36$$

$$y_{p_{1}} = 6 \qquad x^{2}(0) - 4x(0) + 6(6) = 36$$

$$y_{p_{1}} = 0 \qquad 36 = 36$$

$$y_{p_{1}} = 0 \qquad 36 = 36$$

$$y_{p_{1}} = 0 \qquad y_{p_{1}} = 6 \quad \text{does solve this equation}$$

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Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) Part 2 Verify that

$$y_{p_{2}} = -7x \text{ solves } x^{2}y'' - 4xy' + 6y = -14x.$$
Substitute
$$x^{2}y_{p_{2}}^{\mu} - 4xy_{p_{2}}^{\mu} + 6y_{p_{2}}^{\mu} = -14x.$$

$$y_{p_{2}}^{\mu} = -7x$$

$$x^{2}(0) - 4x(-7) + 6(-7x) = -14x$$

$$y_{p_{2}}^{\mu} = -7$$

$$28x - 42x = -14x$$

$$-14x = -14x$$

YPZ Solves the ODE

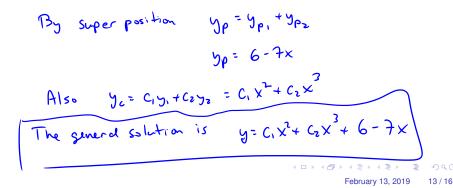
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Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) **Part 3** We already know that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.



Solve the IVP

$$x^{2}y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

We know that the general solution to the ODE is

$$y = C_{1}x^{2} + C_{2}x^{3} + 6 - 7x$$

$$y' = a(_{1}x + 3c_{2}x^{2} - 7)$$

Apply the I.C.

$$y(1) = C_{1}|^{2} + C_{2}|^{3} + 6 - 7(1) = 0$$

$$C_{1} + C_{2} = 0$$

$$C_{1} + C_{2} = 0$$

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$$y'(1) = \partial \zeta_{1}(1) + 3\zeta_{2}1^{2} - 7 = -5$$

$$Q\zeta_{1} + 3\zeta_{2} - 7 = -5$$

$$Q\zeta_{1} + 3\zeta_{2} = Q$$

Solve $\zeta_{1} + \zeta_{2} = 1$

$$\partial \zeta_{1} + 3\zeta_{2} = 2$$

$$Q\zeta_{1} + 3\zeta_{2} = 2$$

$$Q\zeta_{1} + 3\zeta_{2} = 2$$

$$-\zeta_{2} = 0$$

$$\zeta_{1} + 0 = 1$$

The solution to the IVP is

$$y = \chi^{2} + \zeta_{0} - 7\chi$$

$$Q\zeta_{1} + 0 = 1$$

$$\zeta_{1} = 0$$

$$\zeta_{1} + 0 = 1$$

$$\zeta_{2} = 0$$

$$\zeta_{1} = 0$$

$$\zeta_{1} = 0$$

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$$\zeta_{1} = 0$$

$$\zeta_{2} = 0$$

$$\zeta_{1} = 0$$

$$\zeta_{1} = 0$$

$$\zeta_{2} = 0$$

$$\zeta_{3} = 0$$

$$\zeta_{4} = 0$$

$$\zeta_{5} = 0$$

$$\zeta_$$