February 15 Math 2306 sec. 60 Spring 2019

Section 6: Linear Equations Theory and Terminology

We defined **linear dependence** and the Wronskian last time. The following theorem provides a test for linear dependence (or independence).

Theorem (a test for linear independence) Let $f_1, f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

Determine if the functions are linearly dependent or independent:

$$y_1 = x^2, \quad y_2 = x^3 \quad I = (0, \infty)$$

We computed the Wronskian and found

$$W(y_1, y_2)(x) = x^4$$

which is not the zero function. So the functions are linearly **independent**.

Fundamental Solution Set

We're still considering this equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

with the assumptions $a_n(x) \neq 0$ and $a_i(x)$ are continuous on I.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,
(ii) there are n of them, and
(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of n^{th} order Linear Homogeneous Equation

Let y_1, $y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = x^2$ and $y_2 = x^3$ form a fundamental solution set of the ODE

$$x^2y'' - 4xy' + 6y = 0 \quad \text{on} \quad (0, \infty),$$

and determine the general solution.

We want to show that
- there are 2 functions in our set
- they are solutions
- they are linearly independent

There are two of them, y_1 and y_2. We found that $W(y_1, y_2)(x) = x^4 \neq 0$ so they are linearly independent.

we have to verify that the solve the ODE $x^2y'' - 4xy' + 6y = 0$

Check $y_1 = x^2$

$y_1' = 2x$

$y_1'' = 2$

$x^2y_1'' - 4xy_1' + 6y_1 =$

$x^2(2) - 4x(2x) + 6x^2 =$

$2x^2 - 8x^2 + 6x^2 = 0$ y_1 is a solution

Check y_2: $y_2 = x^3$

$y_2' = 3x^2$

$y_2'' = 6x$

$x^2y_2'' - 4xy_2' + 6y_2 =$

$x^2(6x) - 4x(3x^2) + 6x^3 =$

$6x^3 - 12x^3 + 6x^3 = 0$ y_2 is also a solution

The pair does form a fundamental solution set.

The general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$\boxed{y = c_1 x^2 + c_2 x^3}$$

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

The left side is exactly the same as the nonhomogeneous version.

Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \dots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where c_1, $c_2, \dots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$! (complementary plus particular)
The complemenatry part

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

Another Superposition Principle (for nonhomogeneous eqns.)

Let y_{p_1}, y_{p_2}, ..., y_{p_k} be k particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

for $i = 1, \ldots, k$. Assume the domain of definition for all k equations is a common interval I.

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

$g_1(x)$ $g_2(x)$

We will construct the general solution by considering sub-problems.

(a) **Part 1** Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

Substitute

$y_{p_1} = 6$

$y_{p_1}' = 0$

$y_{p_1}'' = 0$

$x^2 y_{p_1}'' - 4x y_{p_1}' + 6y_{p_1} \overset{?}{=} 36$

$x^2(0) - 4x(0) + 6(6) = 36$

$36 = 36$ ✓

$y_{p_1} = 6$ does solve this equation

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) **Part 2** Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

Substitute

$y_{p_2} = -7x$

$y_{p_2}' = -7$

$y_{p_2}'' = 0$

$$x^2 y_{p_2}'' - 4x\, y_{p_2}' + 6y_{p_2} \stackrel{?}{=} -14x$$

$$x^2(0) - 4x(-7) + 6(-7x) \stackrel{?}{=} -14x$$

$$28x - 42x \stackrel{?}{=} -14x$$

$$-14x = -14x \quad \checkmark$$

y_{p_2} solves the ODE

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) **Part 3** We already know that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

$y_{p_1} = 6 \qquad y_{p_2} = -7x$

By superposition $\quad y_p = y_{p_1} + y_{p_2}$

$y_p = 6 - 7x$

Also $\quad y_c = c_1 y_1 + c_2 y_2 = c_1 x^2 + c_2 x^3$

The general solution is $\quad \boxed{y = c_1 x^2 + c_2 x^3 + 6 - 7x}$

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

We know that the general solution to the ODE is

$$y = c_1 x^2 + c_2 x^3 + 6 - 7x$$

$$y' = 2c_1 x + 3c_2 x^2 - 7$$

Apply the I.C.

$$y(1) = c_1 1^2 + c_2 1^3 + 6 - 7(1) = 0$$

$$c_1 + c_2 - 1 = 0$$

$$c_1 + c_2 = 1$$

$$y'(1) = 2c_1(1) + 3c_2 1^2 - 7 = -5$$

$$2c_1 + 3c_2 - 7 = -5$$

$$2c_1 + 3c_2 = 2$$

Solve $\left.\begin{aligned} c_1 + c_2 &= 1 \\ 2c_1 + 3c_2 &= 2 \end{aligned}\right\} \Rightarrow$

$$\begin{aligned} & 2c_1 + 2c_2 = 2 \\ - \quad & 2c_1 + 3c_2 = 2 \\ \hline & -c_2 = 0 \\ & c_2 = 0 \end{aligned}$$

$$c_1 + 0 = 1 \qquad c_1 = 1$$

$$\boxed{\text{The solution to the IVP is} \quad y = x^2 + 6 - 7x}$$