February 15 Math 3260 sec. 55 Spring 2018

Section 2.1: Matrix Operations

Matrix Multiplication We wish to define matrix multiplication in such a way as to correspond to **function composition.** That is, for linear transformations S and T, if

$$S(\mathbf{x}) = B\mathbf{x}, \quad \text{and} \quad T(\mathbf{v}) = A\mathbf{v},$$

then

$$(T \circ S)(\mathbf{x}) = T(S(\mathbf{x})) = A(B\mathbf{x}) = (AB)\mathbf{x}.$$

Matrix Multiplication

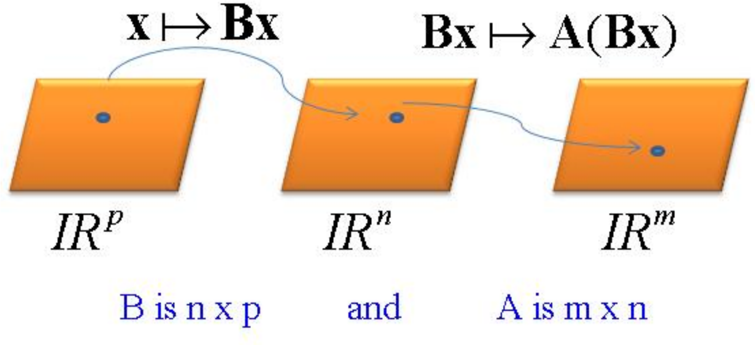

Figure: Composition requires the number of rows of B match the number of columns of A. Otherwise the product is **not defined**.

Matrix Multiplication

$$S : \mathbb{R}^p \longrightarrow \mathbb{R}^n \quad \Longrightarrow \quad B \sim n \times p$$

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \Longrightarrow \quad A \sim m \times n$$

$$T \circ S : \mathbb{R}^p \longrightarrow \mathbb{R}^m \quad \Longrightarrow \quad AB \sim m \times p$$

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p \Longrightarrow$$

$$A(B\mathbf{x}) = x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \cdots + x_pA\mathbf{b}_p \Longrightarrow$$

$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p]$$

The j^{th} column of AB is A times the j^{th} column of B.

Example

Compute the product AB where

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

$$2\times 2 \longleftarrow \text{match} \longrightarrow 2\times 3$$

$$\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 0 \\ -4 \end{bmatrix}, \quad \vec{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$A\vec{b}_1 = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-3 \\ -4+2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$A\vec{b}_2 = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 0-3(-4) \\ 0-8 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

$$A\vec{b}_3 = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2-18 \\ -4+12 \end{bmatrix} = \begin{bmatrix} -16 \\ 8 \end{bmatrix}$$

so

$$AB = \begin{bmatrix} -1 & 12 & -16 \\ -2 & -8 & 8 \end{bmatrix}$$

$$\underset{2\times 2}{A} \quad \underset{2\times 3}{B} \quad \Rightarrow \quad \underset{2\times 3}{AB}$$

Row-Column Rule for Computing the Matrix Product

Suppose A is $m \times n$, and B is $n \times p$. If $AB = C = [c_{ij}]$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

(The ij^{th} entry of the product is the *dot product* of i^{th} row of A with the j^{th} column of B.)

$n = 2$

For example: $AB = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -16 \\ -2 & -8 & 8 \end{bmatrix}$

$$c_{11} = \sum_{k=1}^{2} a_{1k} b_{k1} = a_{11} b_{11} + a_{12} b_{21} = 1(2) + (-3)(1) = -1$$

$$c_{12} = \sum_{k=1}^{2} a_{1k} b_{k2} = a_{11} b_{12} + a_{12} b_{22} = 1 \cdot 0 + (-3)(-4) = 12$$

$$\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

$$c_{13} = \sum_{k=1}^{2} a_{1k} b_{k3} = a_{11} b_{13} + a_{12} b_{23}$$

$$= 1 \cdot 2 + (-3) \cdot 6 = -16$$

$$c_{21} = \sum_{k=1}^{2} a_{2k} b_{k1} = a_{21} b_{11} + a_{22} b_{21}$$

$$= -2 \cdot 2 + 2 \cdot 1 = -2$$

$$c_{22} = \sum_{k=1}^{2} a_{2k} b_{k2} = a_{21} b_{12} + a_{22} b_{22}$$

$$= -2 \cdot 0 + 2 \cdot (-4) = -8$$

$$c_{23} = \sum_{k=1}^{2} a_{2k} b_{k3} = a_{21} b_{13} + a_{22} b_{23}$$

$$= -2 \cdot 2 + 2 \cdot 6 = 8$$

Theorem: Properties-Matrix Product

Let A be an $m \times n$ matrix. Let r be a scalar and B and C be matrices for which the indicated sums and products are defined. Then

(i) $A(BC) = (AB)C$

(ii) $A(B + C) = AB + AC$

(iii) $(B + C)A = BA + CA$

(iv) $r(AB) = (rA)B = A(rB)$, and

(v) $I_m A = A = A I_n$

Caveats!

(1) Matrix multiplication **does not** commute! In general $AB \neq BA$

(2) The zero product property **does not** hold! That is, if $AB = O$, one **cannot** conclude that one of the matrices A or B is a zero matrix.

(3) There is no *cancelation law*. That is, $AB = CB$ **does not** imply that A and C are equal.

Compute AB and BA where $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$.

Both products are defined.

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 6 \end{bmatrix}$$

$$AB \neq BA$$

$$BA = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 11 \\ -1 & 4 \end{bmatrix}$$

Compute the products AB, CB, and BB where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$CB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BB = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$AB = CB$

but

$A \neq C$

$BB = 0$

But $B \neq 0$

Matrix Powers

If A is square—meaning A is an $n \times n$ matrix for some $n \geq 2$, then the product AA is defined. For positive integer k, we'll define

$$A^k = AA^{k-1}.$$

We define $A^0 = I_n$.

So $AA = A^2$, $A^3 = AA^2$ and so on.

Transpose

Definition: Let $A = [a_{ij}]$ be an $m \times n$ matrix. The **transpose** of A is the $n \times m$ matrix denoted and defined by

$$A^T = [a_{ji}].$$

For example, if

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \quad \text{then} \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}.$$

Example

$$A = \begin{bmatrix} 5 & 5 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix}$$

2×2 $\qquad$ 2×3

Compute A^T, B^T, the transpose of the product $(AB)^T$, and the product $B^T A^T$.

$$A^T = \begin{bmatrix} 5 & -1 \\ 5 & 4 \end{bmatrix} \qquad B^T = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 3 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 5 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 35 \\ -6 & 4 & 13 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 5 & -6 \\ 5 & 4 \\ 35 & 13 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 5 & 4 \\ 35 & 13 \end{bmatrix}$$

$3\times 2 \quad 2\times 2$

Note $(AB)^T = B^T A^T$

Theorem: Properties-Matrix Transposition

Let A and B be matrices such that the appropriate sums and products are defined, and let r be a scalar. Then

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(rA)^T = rA^T$

(iv) $(AB)^T = B^T A^T$