February 16 Math 1190 sec. 62 Spring 2017

Section 2.4: Differentiating a Product or Quotient; Higher Order Derivatives

Theorem: (Product Rule) Let f and g be differentiable functions of x. Then the product $f(x)g(x)$ is differentiable. Moreover

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Theorem (Quotient Rule) Let f and g be differentiable functions of x. Then on any interval for which $g(x) \neq 0$, the ratio $\frac{f(x)}{g(x)}$ is differentiable. Moreover

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Example

$$\frac{d}{dx}\frac{f}{g} = \frac{f'g - fg'}{g^2}$$

Evaluate $\quad \dfrac{d}{dx}\left(\dfrac{e^x}{x^2+2x}\right)$

$$= \frac{\left(\frac{d}{dx}e^x\right)(x^2+2x) - e^x\left(\frac{d}{dx}(x^2+2x)\right)}{(x^2+2x)^2}$$

$$= \frac{e^x(x^2+2x) - e^x(2x+2\cdot 1)}{(x^2+2x)^2} = \frac{e^x x^2 + e^x(2x) - e^x(2x) - e^x(2)}{(x^2+2x)^2}$$

$$= \frac{e^x(x^2-2)}{(x^2+2x)^2}$$

Question

$$\frac{d}{dx}\frac{f}{g} = \frac{f'g - fg'}{g^2}$$

Evaluate $f'(x)$ where $f(x) = \dfrac{3x+4}{x^2+1}$

$$f'(x) = \frac{3(x^2+1) - (3x+4)(2x)}{(x^2+1)^2}$$

$$= \frac{3x^2+3 - (6x^2+8x)}{(x^2+1)^2}$$

$$= \frac{3x^2+3-6x^2-8x}{(x^2+1)^2} = \frac{-3x^2-8x+3}{(x^2+1)^2}$$

(a) $f'(x) = \dfrac{3x^2+8x-3}{(x^2+1)^2}$

(b) $f'(x) = \dfrac{3-2x(3x+4)}{(x^2+1)}$

(c) $f'(x) = \dfrac{-3x^2-8x+3}{(x^2+1)^2}$

(d) $f'(x) = \dfrac{-3x^2-8x+3}{x^4+1}$

Higher Order Derivatives:

Given $y = f(x)$, the function f' may be differentiable as well. We may take its derivative which is called the **second derivative** of f. We use the following notation and language:

First derivative: $\dfrac{dy}{dx} = y' = f'(x)$

Second derivative: $\dfrac{d}{dx}\dfrac{dy}{dx} = \dfrac{d^2y}{dx^2} = y'' = f''(x)$

Third derivative: $\dfrac{d}{dx}\dfrac{d^2y}{dx^2} = \dfrac{d^3y}{dx^3} = y''' = f'''(x)$

Fourth derivative: $\dfrac{d}{dx}\dfrac{d^3y}{dx^3} = \dfrac{d^4y}{dx^4} = y^{(4)} = f^{(4)}(x)$

n^{th} derivative: $\dfrac{d}{dx}\dfrac{d^{n-1}y}{dx^{n-1}} = \dfrac{d^ny}{dx^n} = y^{(n)} = f^{(n)}(x)$

Remarks on Notation

- $\frac{d}{dx}$ can *operate* on a function to produce a new function; e.g.

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

- It's too hard to read multiple primes (say beyond 3). Parentheses **must** be used to distinguish powers from derivatives.

$$y^5 \text{ is the fifth power of } y;$$

$$y^{(5)} \text{ is the fifth derivative of } y$$

Example

Compute the first, second, and third derivatives of $f(x) = 3x^4 + 2x^2$.

$$f'(x) = 3(4x^3) + 2(2x) = 12x^3 + 4x$$

$$f''(x) = 12(3x^2) + 4(1) = 36x^2 + 4$$

$$f'''(x) = 36(2x) + 0 = 72x$$

- Note, to get f'' we need f' first, and to get f''' we need f'' first.

Example

Evaluate $F''(x)$ and $F''(2)$ where $F(x) = x^3e^x$.

To get F'', we need F' first. F is a product.

We'll use the product rule. $\frac{d}{dx}fg = f'g + fg'$

$$F'(x) = \left(\frac{d}{dx}x^3\right)e^x + x^3\left(\frac{d}{dx}e^x\right)$$

$$= 3x^2e^x + x^3e^x$$

$$F''(x) = \left(\frac{d}{dx}3x^2\right)e^x + 3x^2\left(\frac{d}{dx}e^x\right) + \left(\frac{d}{dx}x^3\right)e^x + x^3\left(\frac{d}{dx}e^x\right)$$

$$F''(x) = 6xe^x + 3x^2e^x + 3x^2e^x + x^3e^x$$

$$= 6xe^x + 6x^2e^x + x^3e^x$$

So $F''(2) = 6\cdot 2e^2 + 6(2^2)e^2 + 2^3e^2$

$$= 12e^2 + 24e^2 + 8e^2$$

$$= 44e^2$$

Question

Let a, b, and c be nonzero constants. If $y = ax^2 + bx + c$, then $\frac{d^3y}{dx^3}$ is

(a) 0

$y' = a(2x) + b \cdot 1 + 0 = 2ax + b$

(b) $2a + b + c$

$y'' = 2a \cdot 1 + 0 = 2a$

$y''' = 0$

(c) $2a$

(d) cannot be determined without knowing the values of a, b, and c.

Recall the Notation

- $\frac{d}{dx}$ can *operate* on a function to produce a new function; e.g.

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

- It's too hard to read multiple primes (say beyond 3). Parentheses **must** be used to distinguish powers from derivatives.

$$y^5 \text{ is the fifth power of } y;$$

$$y^{(5)} \text{ is the fifth derivative of } y$$

Question

True or False: The fourth derivative of a function $y = f(x)$ is denoted by

$$\frac{dy^4}{dx^4}.$$

it would be $\frac{d^4y}{dx^4}$

Rectilinear Motion

If the position s of a particle in motion (relative to an origin) is a differentiable function $s = f(t)$ of time t, then the derivatives are physical quantities.

Velocity: is the rate of change of position with respect to time. But we know that the derivative is the *rate of change*! Hence the velocity v is the derivative of position. That is,

$$v = \frac{ds}{dt} = f'(t).$$

Rectilinear Motion

Acceleration: is the rate of change of velocity with respect to time. Again, we have a rate of change! The acceleration a is the derivative of the velocity. Thus,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t).$$

Galileo's Law

Galileo's law states that in a vacuum (i.e. in the absence of fluid drag), the position of any object falling near the Earth's surface, subject only to gravity, is proportional to the square of the time elapsed. Mathematically, position s satisfies

$$s = -ct^2.$$

Show that this statement is equivalent to saying that the acceleration due to gravity is constant.

velocity $v = \frac{ds}{dt} = -c(2t) = -2ct$

acceleration $a = \frac{dv}{dt} = -2c(1) = -2c$ a constant!

Question

A particle moves along the x-axis so that its position relative to the origin satisfies $s = t^3 - 4t^2 + 5t$. Determine the acceleration of the particle at time $t = 1$.

$v = \dot{s} = 3t^2 - 8t + 5$

(a) $a(1) = 0$

$a = v' = \ddot{s} = 6t - 8$

(b) $a(1) = -2$

$a(1) = 6 \cdot 1 - 8 = -2$

(c) $a(1) = 6t - 8$

(d) $a(1) = 3t^2 - 8t + 5$

Section 2.5: The Derivative of the Trigonometric Functions

We wish to arrive at derivative rules for each of the six trigonometric functions.

Recall the limits from before

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \quad \text{and} \quad \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0$$

$$\frac{d}{dx}\sin(x) = \cos(x) \quad \text{and} \quad \frac{d}{dx}\cos(x) = -\sin(x)$$

We'll prove the first (the second is left as an exercise).

By definition

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

* From trig

$$\sin(x+h) = \cos(x)\sin(h) + \sin(x)\cos(h)$$

$$= \lim_{h\to 0}\frac{\cos(x)\sin(h)+\sin(x)\cos(h) - \sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos(x)\sin(h) + \sin(x)\left(\cos(h)-1\right)}{h}$$

$$= \lim_{h\to 0}\left(\frac{\cos(x)\sin(h)}{h} + \frac{\sin(x)\left(\cos(h)-1\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\cos(x)\left(\frac{\sin(h)}{h}\right) + \sin(x)\left(\frac{\cos(h)-1}{h}\right)\right)$$

$$= \lim_{h\to 0} \cos(x)\left(\frac{\sin(h)}{h}\right) + \lim_{h\to 0} \sin(x)\left(\frac{\cos(h)-1}{h}\right)$$

$$= \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} + \sin(x) \lim_{h\to 0} \frac{\cos(h)-1}{h}$$

$$= \cos(x)\cdot 1 + \sin(x)\cdot 0$$

$$= \cos(x)$$

$$\boxed{\frac{d}{dx}\sin(x) = \cos(x)}$$

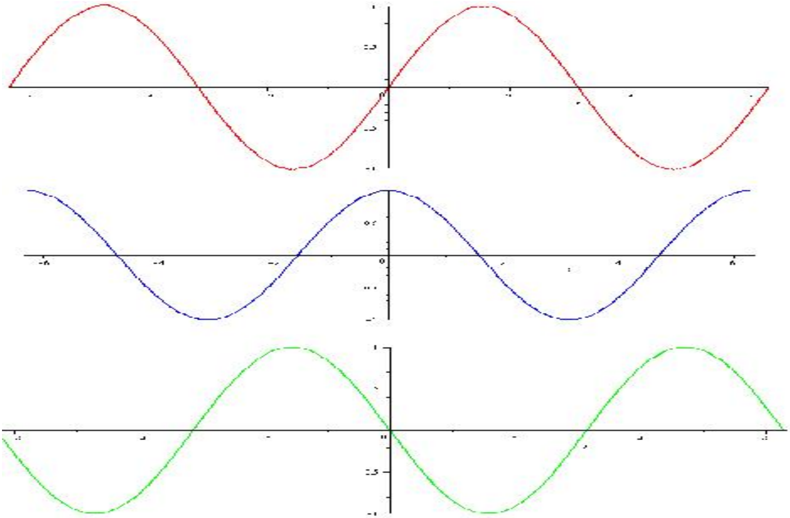

Figure: Graphs of $y = \sin x$, $y = \cos x$, $y = -\sin x$ (from top to bottom).

Example: Evaluate the derivative.

$$\frac{d}{dx}k f(x) = k f'(x)$$

$$\frac{d}{dx}\left(\sin x + 4\cos x\right)$$

$$= \frac{d}{dx}\sin x + 4\frac{d}{dx}\cos x$$

$$= \cos x + 4(-\sin x)$$

$$= \cos x - 4\sin x$$

Product rule: $\frac{d}{dx}4\cos x = \left(\frac{d}{dx}4\right)\cos x + 4\left(\frac{d}{dx}\cos x\right)$

$$= 0\cdot\cos x + 4(-\sin x) = -4\sin x$$

Question

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x,$$

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

Evaluate the derivative $\frac{d}{dx}x\sin x$ $= \left(\frac{d}{dx}x\right)\sin x + x\left(\frac{d}{dx}\sin x\right)$

(a) $\sin x + \cos x$

(b) $x\cos x - \sin x$

(c) $\sin x + x\cos x$

(d) $1 \cdot \cos x$

Use the fact that $\tan x = \sin x / \cos x$ to determine the derivative rule for the tangent.

$$\frac{d}{dx}\frac{f}{g} = \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\left(\frac{d}{dx}\sin x\right)\cos x - \sin x\left(\frac{d}{dx}\cos x\right)}{(\cos x)^2}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

Six Trig Function Derivatives

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\cot x = -\csc^2 x,$$

$$\frac{d}{dx}\sec x = \sec x\tan x, \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$

Question

Which of the following is correct?

(a) $\dfrac{d}{dx}e^x = xe^{x-1}$

(b) $\dfrac{d}{dx}e^x = e^x$

(c) $\dfrac{d}{dx}e^x = 0$ since e is constant.

Question

If $g(t) = 2e^t - \cot(t)$, then

(a) $\dfrac{dg}{dt} = 2e^t - \dfrac{\cos t}{\sin t}$

(b) $\dfrac{dg}{dt} = 2te^{t-1} + \csc^2 t$

(c) $\dfrac{dg}{dt} = 2e^t + \csc^2 t$

(d) $\dfrac{dg}{dt} = 2e^t - \tan t$

Example

Find the equation of the line tangent to the graph of $y = \sec x$ at the point $(\pi/3, 2)$.

We need the slope. The slope

$$m_{tan} = y'\left(\frac{\pi}{3}\right)$$

$$y' = \sec x \tan x \Rightarrow m_{tan} = \sec\frac{\pi}{3}\tan\frac{\pi}{3} = 2\sqrt{3}$$

$$y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$$

$$\boxed{y = 2\sqrt{3}\,x - \frac{2\sqrt{3}\,\pi}{3} + 2}$$

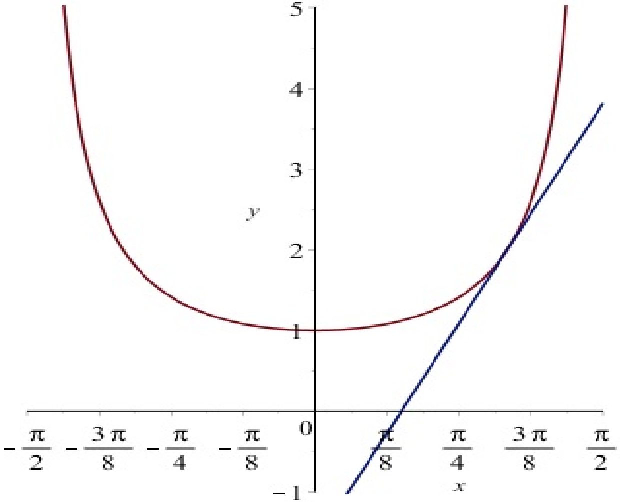

Figure: Graphs of $y = \sec x$ and $y = 2\sqrt{3}x - \frac{2\sqrt{3}\pi}{3} + 2$