February 16 Math 2306 sec 58 Spring 2016

Section 6: Linear Equations Theory and Terminology

We're still considering this equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

with the assumptions $a_n(x) \neq 0$ and $a_i(x)$ are continuous on I.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,
(ii) there are n of them, and
(iii) they are linearly independent.

General Solution of n^{th} order Linear Homogeneous Equation

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Consider $x^2y'' - 4xy' + 6y = 0$ for $x > 0$

Determine which if any of the following sets of functions is a fundamental solution set.

(a) $y_1 = 2x^2, \quad y_2 = x^2$

(b) $y_1 = x^2, \quad y_2 = x^{-2}$

(c) $y_1 = x^3, \quad y_2 = x^2$

(d) $y_1 = x^2, \quad y_2 = x^3, \quad y_3 = x^{-2}$

We determined that (a) was linearly dependent, (d) has the wrong number of potential solutions, and $y_2 = x^{-2}$ from set (b) doesn't solve the ODE. The function $y_2 = x^2$ from set (c) DOES solve the ODE.

Does $y_1 = x^3$ solve the ODE?

$y_1 = x^3$ $\qquad$ $x^2 y_1'' - 4x y_1' + 6y_1 =$

$y_1' = 3x^2$ $\qquad$ $x^2(6x) - 4x(3x^2) + 6x^3 =$

$y_1'' = 6x$ $\qquad$ $6x^3 - 12x^3 + 6x^3 = 0$

Yes, it's a solution.

Check linear independence.

$$W(y_1, y_2)(x) = \begin{vmatrix} x^3 & x^2 \\ 3x^2 & 2x \end{vmatrix}$$

$$= x^3(2x) - 3x^2(x^2) = 2x^4 - 3x^4 = -x^4$$

$$W(y_1, y_2)(x) = -x^4 \neq 0$$

They are linearly independent.

So $y_1 = x^3$, $y_2 = x^2$ is a fundamental solution set to $x^2y'' - 4xy' + 6y = 0$ on $x > 0$.

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Write the associated homogeneous equation

(a) $x^3y''' - 2x^2y'' + 3xy' + 17y = e^{2x}$

$$x^3y''' - 2x^2y'' + 3xy' + 17y = 0$$

(b) $\dfrac{d^2y}{dx^2} + 14\dfrac{dy}{dx} = \cos\left(\dfrac{\pi x}{2}\right)$

$$\frac{d^2y}{dx^2} + 14\frac{dy}{dx} = 0$$

Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{y_c} + y_p(x)$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)

Another Superposition Principle (for nonhomogeneous eqns.)

Let y_{p_1}, y_{p_2}, ..., y_{p_k} be k particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

for $i = 1, \ldots, k$. Assume the domain of definition for all k equations is a common interval I.

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

$$\underbrace{36}_{g_1(x)} \quad \underbrace{-14x}_{g_2(x)}$$

(a) Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

$y_{p_1} = 6$

$y_{p_1}' = 0$

$y_{p_1}'' = 0$

$$x^2 y_{p_1}'' - 4x y_{p_1}' + 6y_{p_1} =$$

$$x^2(0) - 4x(0) + 6(6) =$$

$$36 = 36$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

$y_{p_2} = -7x$

$y_{p_2}' = -7$

$y_{p_2}'' = 0$

$x^2 y_{p_2}'' - 4x y_{p_2}' + 6y_{p_2} =$

$x^2(0) - 4x(-7) + 6(-7x) =$

$28x - 42x =$

$-14x = -14x$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) Recall that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

From (a) and (b) $y_p = y_{p_1} + y_{p_2} = 6 - 7x$

The general solution is

$$\boxed{y = \underbrace{c_1 x^2 + c_2 x^3}_{y_c} + \underbrace{6 - 7x}_{y_p}}$$

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

The general solution to the ODE is

$y = c_1x^2 + c_2x^3 + 6 - 7x$, $y(1) = c_1 1 + c_2 \cdot 1 + 6 - 7 \cdot 1 = 0$

$y' = 2c_1x + 3c_2x^2 - 7$, $y'(1) = 2c_1 \cdot 1 + 3c_2 \cdot 1 - 7 = -5$

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ 2c_1 + 3c_2 &= 2 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} 2c_1 + 2c_2 &= 2 \\ 2c_1 + 3c_2 &= 2 \\ \hline -c_2 &= 0 \end{aligned}$$

subtract

$C_2 = 0$ so $C_1 = 1$

The solution to the IVP is

$$y = x^2 + 6 - 7x \, .$$

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

where $P = a_1/a_2$ and $Q = a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1y_1(x) + c_2y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

$$y_2(x) = u(x)y_1(x)$$

for some function $u(x)$. The method involves finding the function u.

Example

Verify that $y_1 = e^{-x}$ is a solution of $y'' - y = 0$. Then find a second solution y_2 of the form

$$y_2(x) = u(x)y_1(x) = e^{-x}u(x).$$

Confirm that the pair y_1, y_2 is linearly independent.

Verify y_1 solves the ODE:

$$y_1 = e^{-x}, \quad y_1' = -e^{-x}, \quad y_1'' = e^{-x}$$

$$y_1'' - y_1 = e^{-x} - e^{-x} = 0$$

Yes y_1 solves the ODE.

$$y'' - y = 0$$

$y_2 = e^{-x} u$ this must solve the ODE

$$y_2' = -e^{-x} u + e^{-x} u'$$

$$y_2'' = e^{-x} u - e^{-x} u' - e^{-x} u' + e^{-x} u''$$

$$= e^{-x} u - 2e^{-x} u' + e^{-x} u''$$

$$y_2'' - y_2 = e^{-x} u - 2e^{-x} u' + e^{-x} u'' - e^{-x} u = 0$$

$$e^{-x}\cancel{u} - 2e^{-x}u' + e^{-x}u'' - \cancel{e^{-x}u} = 0$$

$$e^{-x}(u'' - 2u') = 0$$

$$\Rightarrow \quad u'' - 2u' = 0$$

Let $w = u'$, then $w' = u''$ in w, this is the 1st order equation

$$w' - 2w = 0$$

1st order linear

and separable

$$\frac{dw}{dx} = 2w \implies \frac{1}{w}\,dw = 2\,dx$$

$$\int \frac{1}{w}\,dw = \int 2\,dx = 2x$$

$$\ln w = 2x \implies w = e^{2x}$$

$w = u'$ so $u = \int w\,dx = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$

$$y_2 = u \cdot y_1 = \frac{1}{2}e^{2x} \cdot e^{-x} = \frac{1}{2}e^{x}$$

The general solution is $y = C_1 e^{-x} + C_2 e^{x}$.

For lin. independence, see slides of 2/11/16

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - -\text{is known.}$$

Assume $y_2 = y_1 u$ $\quad u(x)$ - some function TBD

$y_2' = y_1 u' + y_1' u$

$y_2'' = y_1 u'' + y_1' u' + y_1' u' + y_1'' u$

$= y_1 u'' + 2y_1' u' + y_1'' u$

$$y_2'' + P(x)y_2' + Q(x)y_2 =$$

$$y_1u'' + 2y_1'u' + y_1''u + P(x)\left(y_1u' + y_1'u\right) + Q(x)y_1u = 0$$

$$y_1u'' + \left(2y_1' + P(x)y_1\right)u' + \underbrace{\left(y_1'' + P(x)y_1' + Q(x)y_1\right)}u = 0$$

Recall y_1 solves the homogeneous eqn.

i.e. $y_1'' + P(x)y_1' + Q(x)y_1 = 0$

We have

$$y_1 u'' + (2y_1' + P(x)y_1) u' = 0$$

we'll finish next time.