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Section 6: Linear Equations Theory and Terminology

General Solution of n^{th} order Linear Homogeneous Equation

Let $y_1, y_2, \dots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants.

Consider $x^2y'' - 4xy' + 6y = 0$ for $x > 0$

Determine which if any of the following sets of functions is a fundamental solution set.

(a) $y_1 = 2x^2, \quad y_2 = x^2$

(b) $y_1 = x^2, \quad y_2 = x^{-2}$

(c) $y_1 = x^3, \quad y_2 = x^2$

(d) $y_1 = x^2, \quad y_2 = x^3, \quad y_3 = x^{-2}$

We found that option (c) was a fundamental solution set and that the general solution is

$$y = c_1x^3 + c_2x^2.$$

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Write the associated homogeneous equation

(a) $x^3y''' - 2x^2y'' + 3xy' + 17y = e^{2x}$

$$x^3y''' - 2x^2y'' + 3xy' + 17y = 0$$

(b) $\dfrac{d^2y}{dx^2} + 14\dfrac{dy}{dx} = \cos\left(\dfrac{\pi x}{2}\right)$

$$\frac{d^2y}{dx^2} + 14\frac{dy}{dx} = 0$$

Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{y_c} + y_p(x)$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)

Another Superposition Principle (for nonhomogeneous eqns.)

Let y_{p_1}, y_{p_2}, …, y_{p_k} be k particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

for $i = 1, \ldots, k$. Assume the domain of definition for all k equations is a common interval I.

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$

Example $x^2y'' - 4xy' + 6y = \underbrace{36}_{g_1(x)} - \underbrace{14x}_{g_2(x)}$

(a) Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

$y_{p_1} = 6$

$y_{p_1}' = 0$

$y_{p_1}'' = 0$

$x^2 y_{p_1}'' - 4x y_{p_1}' + 6y_{p_1} =$

$x^2(0) - 4x(0) + 6(6) =$

$36 = 36$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

$y_{p_2} = -7x$

$y_{p_2}' = -7$

$y_{p_2}'' = 0$

$x^2 y_{p_2}'' - 4x y_{p_2}' + 6 y_{p_2} =$

$x^2(0) - 4x(-7) + 6(-7x) =$

$28x - 42x =$

$-14x = -14x$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) Recall that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

From superposition $y_p = y_{p_1} + y_{p_2} = 6 - 7x$

The general solution is

$$y = c_1 x^2 + c_2 x^3 + 6 - 7x$$

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

From before, the general solution to the ODE is

$$y = c_1 x^2 + c_2 x^3 + 6 - 7x$$

$$y' = 2c_1 x + 3c_2 x^2 - 7$$

$$y(1) = c_1 \cdot 1 + c_2 \cdot 1 + 6 - 7 \cdot 1 = 0 \quad \Rightarrow \quad c_1 + c_2 = 1$$

$$y'(1) = 2c_1 \cdot 1 + 3c_2 \cdot 1 - 7 = -5 \quad \Rightarrow \quad 2c_1 + 3c_2 = 2$$

$$\left.\begin{aligned} 2c_1 + 2c_2 &= 2 \\ 2c_1 + 3c_2 &= 2 \end{aligned}\right\} \Rightarrow \quad c_2 = 0 \quad \text{so} \quad c_1 = 1$$

subtract

$$-c_2 = 0$$

The solution to the IVP is

$$\boxed{y = x^2 + 6 - 7x.}$$

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

where $P = a_1/a_2$ and $Q = a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

$$y_2(x) = u(x)y_1(x)$$

for some function $u(x)$. The method involves finding the function u.

Example

Verify that $y_1 = e^{-x}$ is a solution of $y'' - y = 0$. Then find a second solution y_2 of the form

$$y_2(x) = u(x)y_1(x) = e^{-x}u(x).$$

Confirm that the pair y_1, y_2 is linearly independent.

Verify that y_1 solves the ODE.

$$y_1 = e^{-x}, \quad y_1' = -e^{-x}, \quad y_1'' = e^{-x} \qquad y_1'' - y_1 = e^{-x} - e^{-x} = 0$$

Set $y_2 = e^{-x}u$ u - some function

$$y_2' = e^{-x}u' - e^{-x}u$$

$$y_2'' = e^{-x}u'' - e^{-x}u' - e^{-x}u' + e^{-x}u = e^{-x}u'' - 2e^{-x}u' + e^{-x}u$$

$$y'' - y = 0$$

$$y_2'' - y_2 = e^{-x}u'' - 2e^{-x}u' + e^{-x}\cancel{u} - e^{-x}\cancel{u} = 0$$

$$e^{-x}u'' - 2e^{-x}u' = 0$$

$$e^{-x}(u'' - 2u') = 0 \implies u'' - 2u' = 0$$

Let $w = u'$ so $w' = u''$. In w, the equation is

$$w' - 2w = 0$$ 1st order linear and separable.

$$\frac{dw}{dx} = 2w \implies \frac{1}{w}\,dw = 2\,dx$$

$$\int \frac{1}{w}\,dw = \int 2\,dx \implies \ln w = 2x \implies w = e^{2x}$$

$w = u'$ so $\quad u = \int w\,dx = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$

And $\quad y_2 = u y_1 = \frac{1}{2}e^{2x} \cdot e^{-x} = \frac{1}{2}e^{x}$

To see that y_1, y_2 are linearly independent.

Note $\quad W(y_1, y_2)(x) = \begin{vmatrix} e^{-x} & \frac{1}{2}e^{x} \\ -e^{-x} & \frac{1}{2}e^{x} \end{vmatrix} =$

$$= e^{-x}\left(\frac{1}{2}e^{x}\right) - \left(-e^{-x}\right)\left(\frac{1}{2}e^{x}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

$W(y_1, y_2)(x) = 1 \neq 0$ They are linearly independent.

$$\boxed{\begin{array}{l}\text{The general solution to the homogeneous}\\ \text{equation is}\\ \qquad y = c_1 e^{-x} + c_2 e^{x}\end{array}}$$

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - -\text{is known.}$$

Assume $y_2 = y_1 u(x)$ for some function $u(x)$.

$y_2 = y_1 u$

$y_2' = y_1 u' + y_1' u$

$y_2'' = y_1 u'' + y_1' u' + y_1' u' + y_1'' u = y_1 u'' + 2y_1' u' + y_1'' u$

$$y_2'' + P(x)y_2' + Q(x)y_2 = y_1u'' + 2y_1'u' + y_1''u + P(x)\left(y_1u' + y_1'u\right) + Q(x)y_1u = 0$$

$$y_1u'' + (2y_1' + P(x)y_1)u' + \left(y_1'' + P(x)y_1' + Q(x)y_1\right)u = 0$$

y_1 solves the homogeneous equation!

i.e. $\quad y_1'' + P(x)y_1' + Q(x)y_1 = 0$

We have to solve

$$y_1u'' + (2y_1' + P(x)y_1)u' = 0$$

Let $w = u'$ so $w' = u''$ and divide by y_1

$$w' + \left(2\,\frac{y_1'}{y_1} + P(x)\right) w = 0$$

$$\frac{dw}{dx} = -\left(2\,\frac{dy_1/dx}{y_1} + P(x)\right) w$$

$$\frac{1}{w}\,dw = -\left(2\,\frac{dy_1}{y_1} + P(x)\,dx\right)$$

$$\int \frac{1}{w}\,dw = -2\int \frac{dy_1}{y_1} - \int P(x)\,dx$$

$$\ln w = -2\ln y_1 - \int P(x)\,dx$$

$$\ln w = \ln y_1^{-2} - \int p(x)\,dx$$

$$e^{\ln w} = e^{\ln y_1^{-2} - \int p(x)\,dx} = y_1^{-2} e^{-\int p(x)\,dx}$$

$$w = \frac{e^{-\int p(x)\,dx}}{y_1^2}$$

$w = u'$ so integrate to get

$$\boxed{u = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx}$$

and $\quad y_2 = y_1(x)\,u(x)$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the general solution of the ODE given one known solution

$$x^2y'' - 3xy' + 4y = 0, \quad y_1 = x^2$$

Assume $x > 0$

Standard form $y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$

$P(x) = \frac{-3}{x}$, $-\int P(x)\,dx = -\int \frac{-3}{x}\,dx = 3\ln x = \ln x^3$

So $e^{-\int P(x)\,dx} = e^{\ln x^3} = x^3$

$$u = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx = \int \frac{x^3}{(x^2)^2}\,dx$$

$$= \int \frac{x^3}{x^4}\,dx = \int \frac{1}{x}\,dx = \ln x$$

So $\quad y_2 = u(x)\,y_1(x)$

$$\boxed{y_2 = x^2 \ln x}$$

The general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 x^2 + c_2 x^2 \ln x .$$