February 17 Math 3260 sec. 51 Spring 2020

Section 2.2: Inverse of a Matrix

Recall that if A is an $n \times n$ matrix, and if there exists an $n \times n$ matrix A^{-1} with the property

$$A^{-1}A = AA^{-1} = I$$

then we call

$$A^{-1} \quad \text{the } \textbf{inverse} \text{ of } A.$$

- If A has an inverse, we say it is nonsingular or invertible.
- Otherwise, we say it is singular or not invertible.

Theorem (2 × 2 case)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The number $ad - bc$ is called the determinant of the matrix A. If the determinant of A is not zero, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If the determinant of A is zero, then A is singular.

Algorithm for finding A^{-1}

To find the inverse of a given matrix A:

- Form the $n \times 2n$ augmented matrix $[A \quad I]$.
- Perform whatever row operations are needed to get the first n columns (the A part) to rref.
- If rref(A) is I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$, and the inverse A^{-1} will be the last n columns of the reduced matrix.
- If rref(A) is NOT I, then A is not invertible.

Remarks: We don't need to know ahead of time if A is invertible to use this algorithm.
If A is singular, we can stop as soon as it's clear that rref$(A) \neq I$.

Examples: Find the Inverse if Possible

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} = A$

Set up augmented matrix $[A \ I]$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad -5R_1 + R_3 \Rightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{bmatrix} \qquad 4R_2 + R_3 \to R_3$$

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right] \quad \begin{array}{l} -4R_3 + R_2 \to R_2 \\ -3R_3 + R_1 \to R_1 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 2 & 0 & 16 & -12 & -3 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right] \quad -2R_2 + R_1 \to R_1$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

A is invertible (a.k.a nonsingular)

and
$$A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

Section 2.3: Characterization of Invertible Matrices

Given an $n \times n$ matrix A, we can think of

- A matrix equation $A\mathbf{x} = \mathbf{b}$;

- A linear system that has A as its coefficient matrix;

- A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$;

- Not to mention things like its **pivots**, its **rref**, the linear dependence/independence of its columns, blah blah blah...

Question: How is this stuff related, and how does being singular or invertible tie in?

Theorem: Suppose A is $n \times n$. The following are equivalent. [1]

(a) A is invertible.

(b) A is row equivalent to I_n.

(c) A has n pivot positions.

(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A are linearly independent.

(f) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one to one.

(g) $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

(j) There exists an $n \times n$ matrix C such that $CA = I$.

(k) There exists an $n \times n$ matrix D such that $AD = I$.

(l) A^T is invertible.

[1] Meaning all are true or none are true.

Theorem: (An inverse matrix is unique.)

Let A and B be $n \times n$ matrices. If $AB = I$, then A and B are both invertible with $A^{-1} = B$ and $B^{-1} = A$.

This tells us that if A has an inverse, it has exactly one inverse.

Let's show that B is invertible.

Suppose $B\vec{x} = \vec{0}$. Multiply both sides by A on the left.

$$AB\vec{x} = A\vec{0}$$

$$I\vec{x} = \vec{0}$$

$$\Longrightarrow \quad \vec{x} = \vec{0}$$

So $B\vec{x} = \vec{0}$ has only the trivial solution.

So B is invertible ((d) $\Rightarrow$ (a)).

Since B^{-1} exists, we can use

$$AB = I$$

multiply on each right side by B^{-1}

$$ABB^{-1} = IB^{-1}$$

$$A = B^{-1}$$

Recall that if a matrix is invertible, its inverse is also invertible; so A is invertible and

$$A = B^{-1},$$

taking the inverse of both sides

$$A^{-1} = (B^{-1})^{-1}$$

$$A^{-1} = B.$$