# February 17 Math 3260 sec. 55 Spring 2020

## Section 2.2: Inverse of a Matrix

Recall that if $A$ is an $n \times n$ matrix, and if there exists an $n \times n$ matrix $A^{-1}$ with the property

$$A^{-1}A = AA^{-1} = I$$

then we call

$$A^{-1} \quad \text{the } \textbf{inverse} \text{ of } A.$$

- If $A$ has an inverse, we say it is nonsingular or invertible.
- Otherwise, we say it is singular or not invertible.

# Theorem ($2 \times 2$ case)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The number $ad - bc$ is called the determinant of the matrix $A$. If the determinant of $A$ is not zero, then $A$ is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If the determinant of $A$ is zero, then $A$ is singular.

# Algorithm for finding $A^{-1}$

To find the inverse of a given matrix $A$:

- Form the $n \times 2n$ augmented matrix $[A \quad I]$.
- Perform whatever row operations are needed to get the first $n$ columns (the $A$ part) to rref.
- If rref$(A)$ is $I$, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$, and the inverse $A^{-1}$ will be the last $n$ columns of the reduced matrix.
- If rref$(A)$ is NOT $I$, then $A$ is not invertible.

**Remarks:** We don't need to know ahead of time if $A$ is invertible to use this algorithm.
If $A$ is singular, we can stop as soon as it's clear that rref$(A) \neq I$.

# Examples: Find the Inverse if Possible

Set up an augmented matrix

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} = A$

$[A \quad I]$

Do row reduction

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \quad -5R_1 + R_3 \to R_3$$

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right] \quad 4R_2 + R_3 \to R_3$$

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right] \quad \begin{array}{l} -3R_3 + R_1 \to R_1 \\ -4R_3 + R_2 \to R_2 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 2 & 0 & 16 & -12 & -3 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right] \quad -2R_2 + R_1 \to R_1$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

$\mathrm{rref}(A) = I$ so $A^{-1}$ exists and

$$A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

# Section 2.3: Characterization of Invertible Matrices

Given an $n \times n$ matrix $A$, we can think of

- A matrix equation $A\mathbf{x} = \mathbf{b}$;
- A linear system that has $A$ as its coefficient matrix;
- A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$;
- Not to mention things like its **pivots**, its **rref**, the linear dependence/independence of its columns, blah blah blah...

**Question:** How is this stuff related, and how does being singular or invertible tie in?

## Theorem: Suppose $A$ is $n \times n$. The following are equivalent. [1]

(a) $A$ is invertible.

(b) $A$ is row equivalent to $I_n$.

(c) $A$ has $n$ pivot positions.

(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of $A$ are linearly independent.

(f) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one to one.

(g) $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of $A$ span $\mathbb{R}^n$.

(i) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

(j) There exists an $n \times n$ matrix $C$ such that $CA = I$.

(k) There exists an $n \times n$ matrix $D$ such that $AD = I$.

(l) $A^T$ is invertible.

[1] Meaning all are true or none are true.

# Theorem: (An inverse matrix is unique.)

Let $A$ and $B$ be $n \times n$ matrices. If $AB = I$, then $A$ and $B$ are both invertible with $A^{-1} = B$ and $B^{-1} = A$.

This says that if $A$ has an inverse, it has only one.

Suppose $AB = I$ and consider the homogeneous equation $B\vec{x} = \vec{0}$. Multiply each side on its left by $A$.

$$AB\vec{x} = A\vec{0}$$

$$I\vec{x} = \vec{0} \implies \vec{x} = \vec{0}.$$

$B\vec{x} = \vec{0}$ has only the trivial solution. Hence $B$ is invertible. ((d) $\Rightarrow$ (a))

$B^{-1}$ exists, so let's multiply $AB = I$ by $B^{-1}$ on the right side.

$$ABB^{-1} = IB^{-1}$$

$$AI = B^{-1}$$

$$A = B^{-1}.$$

Since $B$, hence $B^{-1}$, is invertible, $A$ is invertible. Moreover,

$$A^{-1} = (B^{-1})^{-1} = B.$$