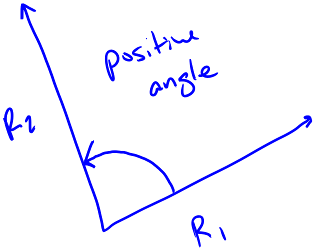
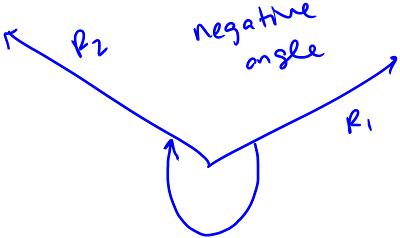
Angles, Rotations, and Angle Measures

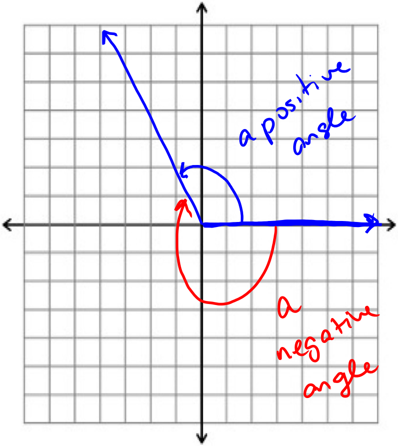
We define an angle by a pair of rays (say R_1 and R_2) that share a common origin. We can indicate direction for an angle by indicating one ray as the **initial ray** and the other as the **terminal ray**.

We then define an angle as being **positive** if it is counter clock-wise and **negative** if it is clock-wise.





Angles in Standard Position



• Vertex at the origin

• initial ray is the +x-axis

• The terminal ray starts at (0,0) and can go into any quadrant or along any axis.

Angles in Standard Position

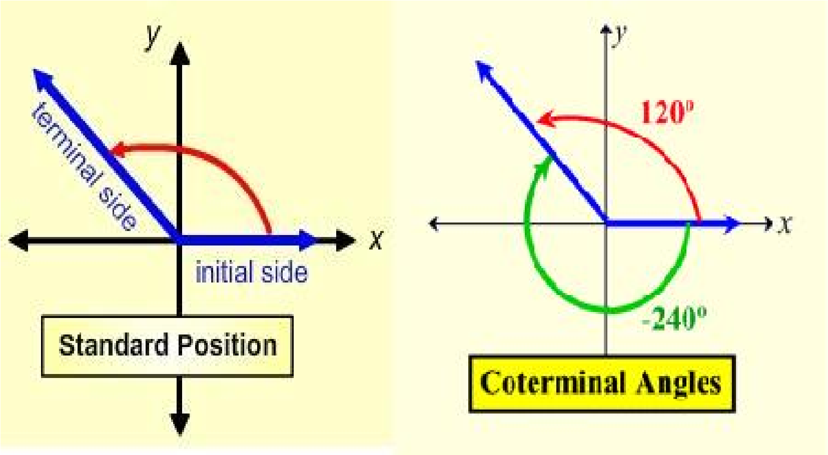


Figure: An angle in STANDARD POSITON has the $+x$-axis as its initial side. More than one angle may have the same terminal side. These are called *co-terminal*.

Degree Measure

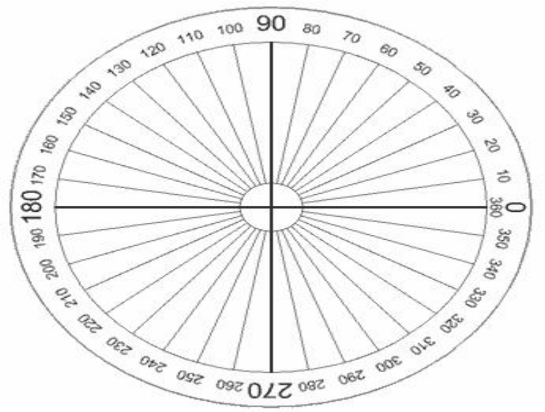
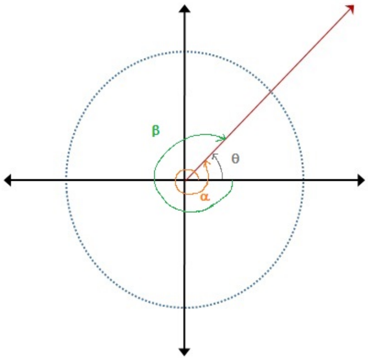


Figure: We can asign a measure to the angle between an initial and terminal side. **Degree** measure is obtained by dividing one full rotation into 360 equal parts.

Coterminal Angles

Since we can measure clockwise (neg.) or counter clockwise (pos.), and can allow for full rotations, angles in standard position may be coterminal but of different measure.



$\alpha = \theta + 360°$

$\beta = \theta - 360°$

Figure: The three angles θ, α, and β have different measures but are coterminal. **Note: Coterminal angles differ by a multiple of** $360°$**.**

Example

Find two angles that are coterminal with 30°. Choose one that is negative, and one whose measure is greater than 360°.

Simple answers would be

$30^\circ - 360^\circ = -330^\circ$ is a negative one

$30^\circ + 360^\circ = 390^\circ$ is one larger than 360°

Question

Recall that: Coterminal angles differ by a multiple of 360°.

Which of the following angles is coterminal with $-45°$?

(a) 45°

(b) 135°

(c) 225°

(d) 315°

$-45^\circ + k(360^\circ)$ for some integer k

Complementary and Supplementary Angles

Definition: Two positive angles whose measures sum to 90° are called **complementary** angles.

Definition: Two positive angles whose measures sum to 180° are called **supplementary** angles.

Example: Find the complementary and the supplementary angles for 38°.

Complement (call it θ) $\theta + 38^\circ = 90^\circ$ $\theta = 52^\circ$

Supplement (call it α) $\alpha + 38^\circ = 180^\circ$ $\alpha = 142^\circ$

Question

Suppose θ is an angle whose measure is between 0° and 90°. The **complementary** angle to θ is

(a) $\theta + 90°$

(b) $\theta - 90°$

(c) $90° - \theta$

$\theta + (90° - \theta) = 90°$

(d) could be any one of the above depending on the actual measure of θ

Radian Measure (Some section 6.4)

Degree measure is sometimes used in technical fields (surveying and engineering). But degrees complicate many mathematical computations. We prefer another measure that is in some sense *unitless* [1].

Radians: (Rad) An angle is measured in radians in relation to a unit circle (circle of radius 1).

An angle $\theta = 1$ radian if the angle subtends an arc in a unit circle of length 1.

[1] We'll still call them units, but it will become more clear that they aren't units in the traditional sense.

A Radian

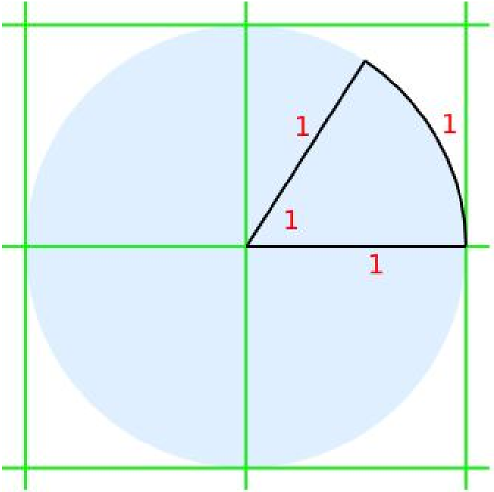


Figure: One Radian: The length of the arc equals the radius of the circle.

Radian Measure

The arc-length of a whole unit circle is 2π. So...

There are 2π radians in one circle (a little more than 6 of them)!

Converting Between Degrees & Radians

Since $360^\circ = 2\pi$ rad, we get the following conversion factors:

$$1^\circ = \frac{\pi}{180} \text{ rad} \qquad \text{and} \qquad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ$$

Remark: If an angle doesn't have the degree symbol $^\circ$ next to it, it is assumed to be in radians!

Converting Between Angle Measures

- To convert from degrees to radians, multiply by

$$\frac{\pi}{180}.$$

- To convert from radians to degrees, multiply by

$$\frac{180}{\pi} \quad \text{and insert the symbol} \quad \circ .$$

Example

Convert each angle measure to the other *units*.

(a) 45°

to radians mult. by $\frac{\pi}{180}$ $\quad 45\left(\frac{\pi}{180}\right)=\frac{45\pi}{180}=\frac{\pi}{4}$

$45^\circ = \frac{\pi}{4}$ rad

(b) $-\frac{\pi}{6}$

to degrees mult by $\frac{180}{\pi}$ add "$^\circ$" $\quad -\frac{\pi}{6}\left(\frac{180}{\pi}\right)=\frac{-180}{6}=-30$

$-\frac{\pi}{6}$ rad $=-30^\circ$

(b) 30

to degrees mult. by $\frac{180}{\pi}$ add "$^\circ$" $\quad 30\left(\frac{180}{\pi}\right)=\frac{5400}{\pi}$

30 rad $=\left(\frac{5400}{\pi}\right)^\circ$

Question

If $\theta = -210^\circ$, then in radians

$$-210 \cdot \frac{\pi}{180} = -\frac{210}{180}\pi = -\frac{7\pi}{6}$$

(a) $\theta = \dfrac{7\pi}{6}$

(b) $\theta = -\dfrac{7\pi}{6}$

(c) $\theta = \dfrac{6\pi}{7}$

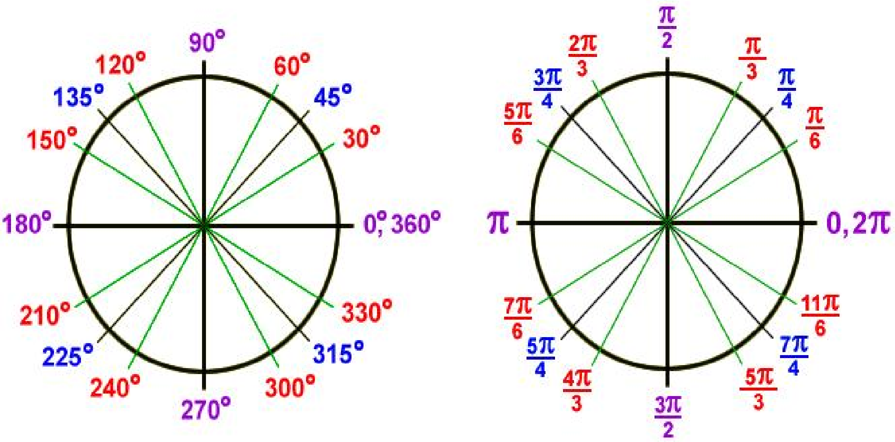
(d) $\theta = -\dfrac{6\pi}{7}$

(e) there's no such thing as a negative angle

Some Common Angles: Degree and Radian

θ°	θ rad
0°	0
30°	$\frac{\pi}{6}$
45°	$\frac{\pi}{4}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
180°	π
270°	$\frac{3\pi}{2}$
360°	2π

Angles With *Nice* Reference Angles



We Recall A Few Terms

Some special names for angles include:

- An **acute** angle is between 0° and 90° (0 and $\frac{\pi}{2}$).
  Terminal side in quadrant I
- An **obtuse** angle is between 90° and 180° ($\frac{\pi}{2}$ and π).
  in quadrant II
- A **right** angle has measure 90° ($\frac{\pi}{2}$).
- A **reflex** angle has measure between 180° (π) and 360° (2π).
  quad III or IV
- **Quadrantal** angles are integer multiples of 90° ($\frac{\pi}{2}$)
- A **straight** angle has measure 180° (π).

(Of course, not all angles fit into one of these categories.)

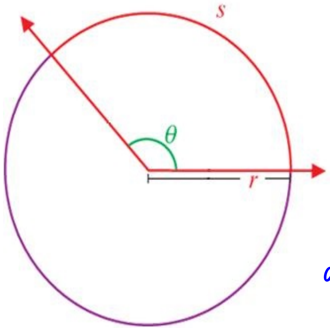
Arclength Formula

Given a circle of radius r, the length s of the arc subtended by the (positive) central angle θ (**in radians**) is given by

vertex @ center

$$s = r\theta.$$

The area of the resulting sector is $A_{sector} = \frac{1}{2}r^2\theta$.



Area of whole circle

$A_{circle} = \pi r^2$

Fraction of circle is $\frac{\theta}{2\pi}$

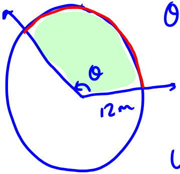
Area of sector is

area of circle × fraction of circle

$A_{sector} = \pi r^2 \times \left(\frac{\theta}{2\pi}\right) = \frac{1}{2} r^2 \theta$

Example

A circle of radius 12 meters has a sector given by a central angle of 135°. Find the associated arc length and the area of the sector.



$\theta = 135°$

arclength $s = r\theta$

area $\frac{1}{2}r^2\theta$

we need θ in radians

$135° = 135\left(\frac{\pi}{180}\right)\text{rad} = \frac{3\pi}{4}\text{ rad}$

$r = 12\text{ m}$ (given)

arclength $s = (12\text{ m})\left(\frac{3\pi}{4}\right) = 9\pi\text{ m}$

area $A_{sector} = \frac{1}{2}(12\text{m})^2\left(\frac{3\pi}{4}\right) = 72\text{m}^2\left(\frac{3\pi}{4}\right) = 54\pi\text{ m}^2$