February 18 Math 2306 sec. 54 Spring 2019

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Let us assume that $a_2(x)\neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

where $P=a_1/a_2$ and $Q=a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

$$y_2(x) = u(x)y_1(x)$$

u can't be constant due to linear independence

for some function $u(x)$. The method involves finding the function u.

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - -\text{is known.}$$

Because y_1 is a solution, we know that

$$y_1'' + P(x)\,y_1' + Q(x)\,y_1 = 0$$

Assume $\quad y_2 = u(x)\,y_1(x) \qquad$ substitute into the ODE

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$$

$$= u''y_1 + 2u'y_1' + uy_1''$$

$$\underline{y_2''} + P(x)\underline{y_2'} + Q(x)\underline{y_2} = 0$$

$$\underline{u''y_1 + 2u'y_1' + uy_1''} + P(x)(\underline{u'y_1 + uy_1'}) + Q(x)\underline{uy_1} = 0$$

Collect u, u', u'' terms

$$u''y_1 + (2y_1' + P(x)y_1)u' + \underbrace{(y_1'' + P(x)y_1' + Q(x)y_1)}_{\overset{\shortparallel}{0} \text{ since } y_1 \text{ solves the ODE}}u = 0$$

$$u''y_1 + (2y_1' + P(x)y_1)u' = 0$$

Let $w = u'$ so that $w' = u''$.

The equation for w is 1^{st} order linear and separable

$$y_1 w' + (2y_1' + P(x)\, y_1)\, w = 0$$

Let's assume that $w > 0$ and separate variables

$$y_1 \frac{dw}{dx} = -\left(2\frac{dy_1}{dx} + P(x)\, y_1\right) w$$

$$\frac{1}{w}\frac{dw}{dx} = -\frac{1}{y_1}\left(2\frac{dy_1}{dx} + P(x) y_1\right)$$

$$\int \frac{1}{w}\, dw = -\int \left(\frac{2}{y_1}\frac{dy_1}{dx} + P(x)\right) dx$$

$$\int \frac{1}{w}\,dw = -\int \frac{2}{y_1}\frac{dy_1}{dx}\,dx - \int P(x)\,dx$$

$$\int \frac{1}{w}\,dw = \int \frac{-2}{y_1}\,dy_1 - \int P(x)\,dx$$

$$\ln w = -2\ln|y_1| - \int P(x)\,dx$$

$$w = e^{\ln y_1^{-2} - \int P(x)dx} = e^{\ln y_1^{-2}} \cdot e^{-\int P(x)dx}$$

$$w = y_1^{-2}\, e^{-\int P(x)dx}$$

$$W = \frac{e^{-\int P(x)dx}}{(y_1)^2}, \quad W = u' \quad \text{so}$$

$$u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2}\, dx$$

and $\quad y_2 = u y_1$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x) \underbrace{\int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx}_{u}$$

Example

Find the general solution of the ODE given one known solution

$$x^2y'' - 3xy' + 4y = 0, \quad y_1 = x^2$$

Let's assume $x > 0$. Standard form

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

$y_2 = uy_1$ where $u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2}\,dx$

$P(x) = \frac{-3}{x}$

for $x > 0$

$$e^{-\int P(x)dx} = e^{-\int \frac{-3}{x}dx} = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = e^{\ln x^3} = x^3$$

and $(y_1)^2 = (x^2)^2 = x^4$

$$u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2}dx = \int \frac{x^3}{x^4}dx = \int \frac{1}{x}\,dx = \ln x$$

So $y_2 = uy_1 = (\ln x)x^2 = x^2 \ln x$

The general solution is

$$y = C_1 x^2 + C_2 x^2 \ln x$$

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

a, b, c constant

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

$a \neq 0$

Question: What sort of function y could be expected to satisfy

$$y'' = \text{constant}\, y' + \text{constant}\, y?$$

$y = e^{mx}$ should work

We look for solutions of the form $y = e^{mx}$ with m constant.

Substitute into $ay'' + by' + cy = 0$

$y = e^{mx}$

$y' = me^{mx}$

$y'' = m^2 e^{mx}$

$$a(m^2 e^{mx}) + b(me^{mx}) + c(e^{mx}) = 0$$

$$e^{mx}(am^2 + bm + c) = 0$$

This holds if m solves

$$am^2 + bm + c = 0$$

am^2+bm+c is called the characteristic polynomial for the ODE and

$$am^2+bm+c=0$$

is the characteristic equation.

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent.

Using the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$$

$$= e^{m_1 x}\left(m_2 e^{m_2 x}\right) - m_1 e^{m_1 x} e^{m_2 x}$$

$$W(y_1, y_2)(x) = (m_2 - m_1)\, e^{(m_1+m_2)x}$$

Claim: $W \neq 0$

Note $m_2 - m_1 \neq 0$ since $m_1 \neq m_2$