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Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Such an equation in **standard form** looks like

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

We are assuming that one solution $y_1(x)$ is known, and we seek a second linearly independent solution y_2 of the form

$$y_2(x) = u(x)y_1(x) \quad \text{for some function } u.$$

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - -\text{is known.}$$

We set $y_2 = y_1 u$ and upon substitution found that u solves the ODE

$$y_1 u'' + (2y_1' + P(x)y_1)u' = 0$$

It remains to solve this equation for u and then write the second solution y_2.

Let $w = u'$ then $w' = u''$, the equation becomes

$$y_1 w' + (2y_1' + P(x)y_1)w = 0$$

1st order linear and separable.

Separating variables

$$w' = -\left(2\frac{y_1'}{y_1} + P(x)\right)w$$

$$\frac{1}{w}\frac{dw}{dx} = -\left(2\frac{dy_1/dx}{y_1} + P(x)\right)$$

$$\frac{1}{w}\frac{dw}{dx}dx = -2\frac{dy_1/dx}{y_1}dx - P(x)\,dx$$

$$\int \frac{1}{w}\,dw = -\int 2\,\frac{dy_1}{y_1} - \int p(x)\,dx$$

$$\ln w = -2\ln y_1 - \int p(x)\,dx$$

$$\ln w = \ln y_1^{-2} - \int p(x)\,dx$$

$$e^{\ln w} = e^{\ln y_1^{-2} - \int p(x)\,dx} = e^{\ln y_1^{-2}} \cdot e^{-\int p(x)\,dx}$$

$$w = y_1^{-2}\, e^{-\int p(x)\,dx}$$

$w = u'$ so $u = \int w\,dx = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$

So $u = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$ and

$$y_2 = y_1(x)\,u(x) = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the general solution of the ODE given one known solution

$$x^2y'' - 3xy' + 4y = 0, \quad y_1 = x^2 \qquad x > 0$$

Standard form $\quad y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$

$P(x) = \frac{-3}{x} \qquad -\int P(x)\,dx = -\int \frac{-3}{x}\,dx = 3\ln x = \ln x^3$

$$u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2}\,dx = \int \frac{e^{\ln x^3}}{(x^2)^2}\,dx$$

$$= \int \frac{x^3}{x^4}\,dx = \int \frac{1}{x}\,dx = \ln x$$

So $\quad y_2 = u y_1 = (\ln x)\,x^2 = x^2 \ln x$

The general solution to the homogeneous eqn is

$$y = c_1 x^2 + c_2 x^2 \ln x$$

Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 1, \quad y'(0) = -2$$

$P(x) = 4 \qquad -\int P(x)\,dx = -\int 4\,dx = -4x$

$$u = \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx = \int \frac{e^{-4x}}{(e^{-2x})^2}\,dx$$

$$= \int \frac{e^{-4x}}{e^{-4x}}\,dx = \int dx = x$$

$y_2 = u y_1 = x e^{-2x}$ The general solution to the ODE is $y = c_1 e^{-2x} + c_2 x e^{-2x}$.

Apply $y(0)=1$, $y'(0)=-2$

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$y'(x) = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x e^{-2x}$$

$$y(0) = c_1 e^0 + c_2 \cdot 0 = c_1 = 1$$

$$y'(0) = -2c_1 e^0 + c_2 e^0 - 2c_2 \cdot 0 = -2$$

$$-2\cdot 1 + C_2 = -2 \Rightarrow \quad C_2 = 0$$

The solution to the IVP is

$$y = e^{-2x}.$$

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

Question: What sort of function y could be expected to satisfy

$$y'' = \text{constant}\, y' + \text{constant}\, y?$$

We look for solutions of the form $y = e^{mx}$ with m constant.

$$ay'' + by' + cy = 0$$

$$y = e^{mx}$$

$$y' = me^{mx}$$

$$y'' = m^2e^{mx}$$

$$ay'' + by' + cy =$$

$$am^2e^{mx} + bme^{mx} + ce^{mx} =$$

$$e^{mx}(am^2 + bm + c) = 0$$

This will be true if

$$am^2 + bm + c = 0$$

So we have a solution e^{mx} if

m solves this quadratic equation.

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent.

Use the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$$

$$= e^{m_1 x}\left(m_2 e^{m_2 x}\right) - m_1 e^{m_1 x} \cdot e^{m_2 x}$$

$$= e^{(m_1+m_2)x}\left(m_2 - m_1\right) \neq 0$$

If it were zero, this would mean

$m_2 - m_1 = 0$ i.e. $m_2 = m_1$

but $m_2 \neq m_1$.