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Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Such an equation in **standard form** looks like

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

We are assuming that one solution $y_1(x)$ is known, and we seek a second linearly independent solution y_2 of the form

$$y_2(x)=u(x)y_1(x) \quad \text{for some function } u.$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 1, \quad y'(0) = -2$$

Use reduction of order to find y_2

$y_2 = u y_1$ where $u = \displaystyle\int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx$

$P(x) = 4$ so $-\int P(x)\,dx = -\int 4\,dx = -4x$

So $$u = \int \frac{e^{-4x}}{(e^{-2x})^2}\,dx = \int \frac{e^{-4x}}{e^{-4x}}\,dx = \int dx = x$$

So $y_2 = u y_1 = x e^{-2x}$

The general solution to the ODE is

$$y = C_1 e^{-2x} + C_2 x e^{-2x}$$

Apply $y(0) = 1$ and $y'(0) = -2$

$$y' = -2C_1 e^{-2x} + C_2 e^{-2x} - 2C_2 x e^{-2x}$$

$$y(0) = C_1 e^0 + C_2 \cdot 0\, e^0 = 1 \implies C_1 = 1$$

$$y'(0) = -2C_1 e^0 + C_2 e^0 - 2C_2 \cdot 0\, e^0 = -2$$

$$-2 + C_2 = -2 \implies C_2 = 0$$

The solution to the IVP is

$$y = e^{-2x}$$

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

Question: What sort of function y could be expected to satisfy

$$y''=\text{constant}\,y'+\text{constant}\,y?$$

We look for solutions of the form $y = e^{mx}$ with m constant.

$$ay'' + by' + cy = 0$$

$$y = e^{mx}$$
$$y' = m e^{mx}$$
$$y'' = m^2 e^{mx}$$

$$ay'' + by' + cy =$$
$$am^2 e^{mx} + bm e^{mx} + c e^{mx} = 0$$
$$e^{mx}(am^2 + bm + c) = 0$$

This will be true if m solves the quadratic equation

$$am^2 + bm + c = 0$$

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent.

Let's compute the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$$

$$= e^{m_1 x}\left(m_2 e^{m_2 x}\right) - m_1 e^{m_1 x} \cdot e^{m_2 x}$$

$$= e^{(m_1+m_2)x}\left(m_2 - m_1\right) \neq 0$$

If $W(y_1,y_2)(x) = 0$, this would require

$m_2 - m_1 = 0$ i.e. $m_2 = m_1$

But $m_1 \neq m_2$ in this case.

Example

Find the general solution of the ODE

$$y'' - 2y' - 2y = 0$$

Characteristic equation: $m^2 - 2m - 2 = 0$

let's complete the square

$$m^2 - 2m - 2 = m^2 - 2m + 1 - 1 - 2$$

$$= (m^2 - 2m + 1) - 3 = 0$$

$$(m-1)^2 = 3 \Rightarrow m - 1 = \pm\sqrt{3}$$

So $\quad m_1 = 1+\sqrt{3}, \quad m_2 = 1-\sqrt{3}$

2 different real roots.

Hence $\quad y_1 = e^{(1+\sqrt{3})x} \quad$ and $\quad y_2 = e^{(1-\sqrt{3})x}$

And the general solution is

$$\boxed{y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}.}$$