February 19 Math 3260 sec. 51 Spring 2020

Section 2.3: Characterization of Invertible Matrices

The following are equivalent:

(a) A is invertible.

(b) A is row equivalent to I_n.

(c) A has n pivot positions.

(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A are linearly independent.

(f) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one to one.

(g) $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

(j) There exists an $n \times n$ matrix C such that $CA = I$.

(k) There exists an $n \times n$ matrix D such that $AD = I$.

(l) A^T is invertible.

Invertible Linear Transformations

Definition: A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that both

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{and} \quad T(S(\mathbf{x})) = \mathbf{x}$$

for every $\mathbf{x}$ in $\mathbb{R}^n$.

If such a function exists, we typically denote it by

$$S = T^{-1}.$$

Theorem (Invertibility of a linear transformation and its matrix)

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and A its standard matrix. Then T is invertible if and only if A is invertible. Moreover, if T is invertible, then

$$T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$$

for every $\mathbf{x}$ in $\mathbb{R}^n$.

Example

Use the standard matrix to determine if the linear transformation is invertible. If it is invertible, characterize the inverse transformation.

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \text{given by} \quad T(x_1, x_2) = (3x_1 - x_2, 4x_2).$$

Calling the standard matrix A,

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2)]$$

$$T(\vec{e}_1) = T(1,0) = (3\cdot 1 - 0,\ 4\cdot 0) = (3,0)$$

$$T(\vec{e}_2) = T(0,1) = (3\cdot 0 - 1,\ 4\cdot 1) = (-1,4)$$

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}$$

The determinant

$$3(4) - (0)(-1) = 12 \neq 0$$

A is invertible and

$$A^{-1} = \frac{1}{12}\begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{12} \\ 0 & \frac{1}{4} \end{bmatrix}$$

T is invertible and $T^{-1}(\vec{x}) = A^{-1}\vec{x}$

Let's put this in the form $T^{-1}(x_1, x_2) = \ldots$

$$A^{-1}\vec{x} = \begin{bmatrix} \frac{1}{3} & \frac{1}{12} \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_1 + \frac{1}{12}x_2 \\ \frac{1}{4}x_2 \end{bmatrix}$$

So

$$T^{-1}(x_1, x_2) = \left(\frac{1}{3}x_1 + \frac{1}{12}x_2, \frac{1}{4}x_2\right)$$

Example

Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a one to one linear transformation. Can we determine whether T is onto? Why (or why not)?

Yes, T is onto.

If A is the standard matrix for T, then $\vec{x} \mapsto A\vec{x}$ being one to one means A is invertible. By our theorem, $\vec{x} \mapsto A\vec{x}$ is onto, so T is also onto.

Section 3.1: Introduction to Determinants

Recall that a 2×2 matrix is invertible if and only if the number called its **determinant** is nonzero. We had

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

We wish to extend the concept of determinant to $n \times n$ matrices in general. And we wish to do so in such a way that invertibility holds if and only if the determinant is nonzero.

For matrix A (square) the determinant can be written

$\det(A)$ or $\det A$ or $|A|$

Determinant 3×3 case:

Suppose we start with a 3×3 invertible matrix. And suppose that $a_{11} \neq 0$. We can multiply the second and third rows by a_{11} and begin row reduction.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \sim$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

Determinant 3×3 case continued...

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

If $A \sim I$, one of the entries in the $2,2$ or the $3,2$ position must be nonzero. Let's assume it is the $2,2$ entry. Continue row reduction to get

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}.$$

Again, if A is invertible, it must be that the bottom right entry is nonzero. That is

$$\Delta \neq 0.$$

Note that if $\Delta = 0$, the rref of A is not I—A would be singular.

Determinant 3×3 case continued...

With a little rearrangement, we have

$$\begin{aligned}\Delta &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + \\ &+ \quad a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}\text{det}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\text{det}\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\text{det}\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\end{aligned}$$

The number Δ will be called the **determinant** of A.

Definitions: Minors

Let $n \geq 2$. For an $n \times n$ matrix A, let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and the j^{th} column of A.

For example, if

$$A = \begin{bmatrix} -1 & 3 & 2 & 0 \\ 4 & 4 & 0 & -3 \\ -2 & 1 & 7 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix} \quad \text{then} \quad A_{23} = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}.$$

Definition: The i, j^{th} **minor** of the $n \times n$ matrix A is the number

$$M_{ij} = \det(A_{ij}).$$

Definitions: Cofactor

Definition: Let A be an $n \times n$ matrix with $n \geq 2$. The i, j^{th} **cofactor** of A is the number

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Example: Find the three minors M_{11}, M_{12}, M_{13} and find the 3 cofactors C_{11}, C_{12}, C_{13} of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

For M_{11}, we need A_{11}

$$A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$\boxed{M_{11} = \det(A_{11}) = a_{22} a_{33} - a_{32} a_{23}}$$

(Example Continued...)

$$C_{11} = (-1)^{1+1} M_{11} = M_{11}$$
$$= a_{22}a_{33} - a_{32}a_{23}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

For M_{12}, we need A_{12}

$$A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{12} = \det(A_{12}) = a_{21}a_{33} - a_{31}a_{23}$$
$$C_{12} = (-1)^{1+2} M_{12} = -(a_{21}a_{33} - a_{31}a_{23})$$

For M_{13}, we need $A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

$$M_{13} = \det(A_{13}) = a_{21}a_{32} - a_{31}a_{22}$$

$$\boxed{C_{13} = (-1)^{1+3} M_{13} = a_{21}\, a_{32} - a_{31}\, a_{22}}$$

Observation:

Comparison with the determinant of the 3×3 matrix, we can note that

$$\Delta = a_{11}\text{det}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\text{det}\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\text{det}\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Definition: Determinant

For $n \geq 2$, the **determinant** of the $n \times n$ matrix $A = [a_{ij}]$ is the number

$$\begin{aligned}\det(A) &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j}\end{aligned}$$

(Well call such a sum a **cofactor expansion.**)

Example: Evaluate the determinant

$$A = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\det(A) = (-1)\det\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix} - 3\det\begin{bmatrix} -2 & 2 \\ 3 & 6 \end{bmatrix} + 0\det\begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix}$$

$$= -1(1\cdot 6 - 0\cdot 2) - 3(-2\cdot 6 - 3\cdot 2) + 0\cdot(-2\cdot 0 - 3\cdot 1)$$

$$= -(6) - 3(-18) + 0 \quad = -6 + 54 = 48$$

Theorem:

The determinant of an $n \times n$ matrix can be computed by cofactor expansion across any row or down any column.

Example: Find the determinant of the matrix

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 \\ 0 & 0 & -3 & 0 \\ -2 & 1 & 2 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix}$$

We can do a cofactor expansion across the 2nd row.

$$\det(A) = \cancel{a_{21}}^{0} C_{21} + \cancel{a_{22}}^{0} C_{22} + a_{23} C_{23} + \cancel{a_{24}}^{0} C_{24}$$

$$= a_{23} C_{23}$$

$$C_{23} = (-1)^{2+3} \det \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix} = (-1)(48) = -48$$

$$\det(A) = -3(-48) = 144$$