February 1 MATH 1112 sec. 54 Spring 2019

Section 5.1: Quick Review of Inverse Functions

Suppose we have the function $y = f(x)$ where $f(x) = x^3 - 1$. We can say

when the input $x = 2$, the output $y = 7$.

We can also say it the other way around

when the output $y = 7$, the input $x = 2$.

Inverse Functions

It doesn't always work so nicely. Consider the example

$$f(x) = x^2.$$

While we can say with confidence

when the input $x = 2$, the output $y = 4$,

We can't be sure what the input is when the output is 4.

Why do I make that claim?

One to One

Definition: A function f is **one to one** if different inputs have different outputs. That is f is one to one provided

$$a \neq b \quad \text{implies} \quad f(a) \neq f(b).$$

Equivalently, f is a one to one function provided

$$f(a) = f(b) \quad \text{implies} \quad a = b.$$

Horizontal Line Test: A function $f(x)$ is one to one if and only if the graph of $y = f(x)$ is intersected at most one time by every horizontal line.

Horizontal Line Test

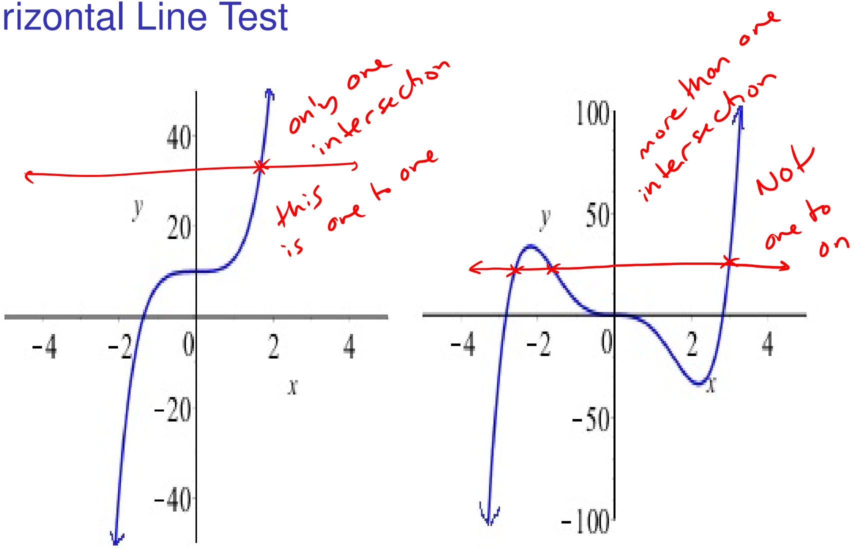
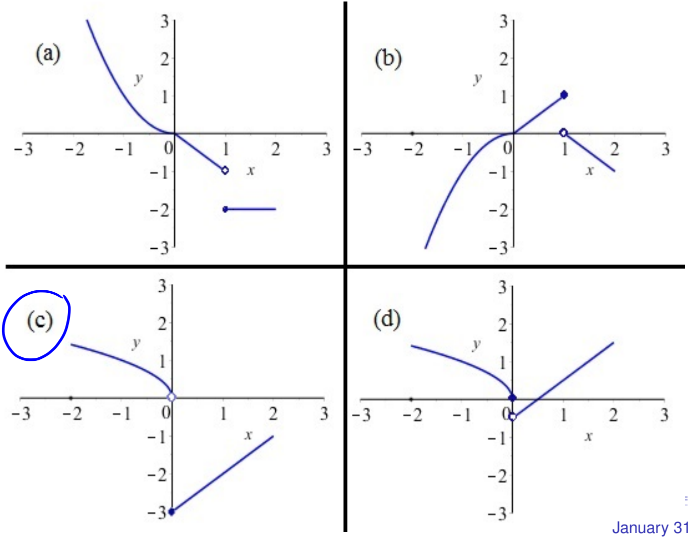


Figure: Left: A one to one function. Right: A function that is not one to one.

Question

Which of the following is the graph of a one to one function? (Hint: Horizontal Line Test)



Inverse Function

Theorem: If f is a one to one function with domain D and range R, then its inverse f^{-1} is a function with domain R and range D. Moreover, the inverse function is defined by

$$f^{-1}(x) = y \quad \text{if and only if} \quad f(y) = x.$$

Characteristic Compositions:
If f is a one to one function with domain D, range R, and with inverse function f^{-1}, then

- for each x in D, $(f^{-1} \circ f)(x) = x$, and
- for each x in R, $(f \circ f^{-1})(x) = x$.

Inverse Function Example

$$f(x) = \frac{1}{x-2} \quad \text{and} \quad f^{-1}(x) = \frac{1+2x}{x}$$

Evaluate $(f \circ f^{-1})(x)$ or $(f^{-1} \circ f)(x)$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{1+2x}{x}\right) = \frac{1}{\left(\frac{1+2x}{x}\right) - 2}$$

$$= \left(\frac{1}{\frac{1+2x}{x} - 2}\right) \cdot \left(\frac{x}{x}\right)$$

Clear fractions multiply by $1 = \frac{x}{x}$

$$= \frac{x}{\left(\frac{1+2x}{x}\right)x - 2x}$$

$$= \frac{x}{1+2x-2x}$$

$$= \frac{x}{1} = x$$

Question

The function $f(x) = \sqrt[3]{x-1}$ is one to one. Which of the following is its inverse function? (Hint: Check compositions $(f^{-1} \circ f)(x)$.)

(a) $f^{-1}(x) = (x+1)^3$

(b) $f^{-1}(x) = (x-1)^3$

(c) $f^{-1}(x) = x^3 + 1$

(d) $f^{-1}(x) = x^3 - 1$

(c)

$f^{-1}(f(x)) = f^{-1}(\sqrt[3]{x-1})$

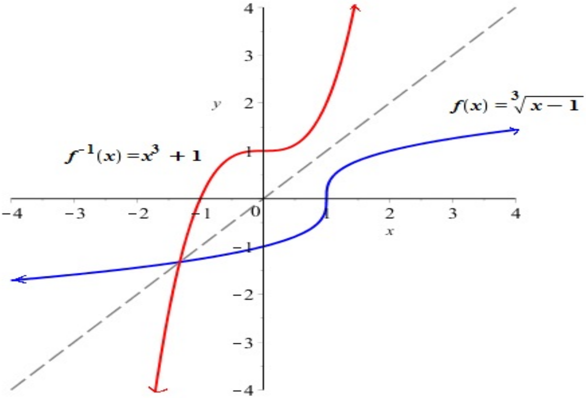
$= (\sqrt[3]{x-1})^3 + 1$

$= (x-1) + 1$

$= x$

Graphs

If (a, b) is a point on the graph of a function, then (b, a) is a point on the graph of its inverse. So the graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.



Restricting the Domain

Sometimes we wish to define an inverse function when a given function is not, in general, one to one. This can be done by restricting the domain of the original function.

The classic example would be the square root. Note $f(x) = x^2$ is not one to one if its domain is $(-\infty, \infty)$. However, if we consider the function $F(x) = x^2$ for $0 \le x < \infty$, this function is one to one with inverse $F^{-1}(x) = \sqrt{x}$.

Restricting the Domain

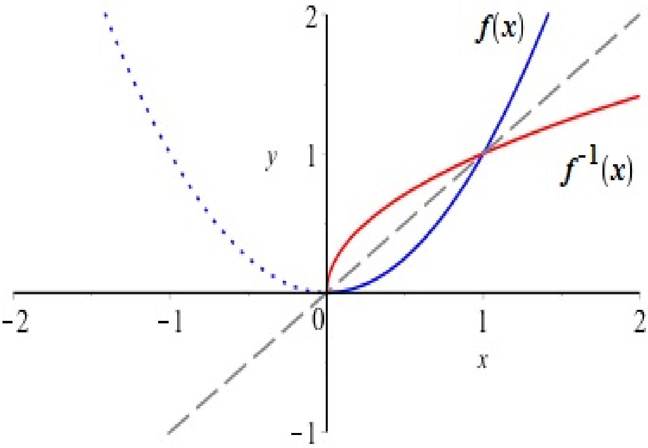


Figure: If the domain of $y = x^2$ is restricted to $[0, \infty)$, the graph passes the horizontal line test.

Section 5.2: Exponential Functions

Many models involve quantities that change at a rate proportional to the quantity itself.

- money in an account with compounded interest,
- bacteria growing in a culture,
- the mass of a substance undergoing radio active decay

These grow or decay in a fashion that is very different from polynomials and rational functions.

The study of how functions change is a big part of Calculus. Here, we will define **exponential** functions and examine some of their properties.

Exponential Functions

Definition: Let a be a positive real number different from 1—i.e. $a > 0$ and $a \neq 1$. The function

$$f(x) = a^x$$

is called the **exponential function of base** a. Its domain is $(-\infty, \infty)$, and its range is $(0, \infty)$.

Some examples of exponential functions are

$$f(x) = 2^x, \qquad g(x) = \left(\frac{1}{3}\right)^x, \quad \text{and} \quad h(x) = \pi^{x-1}$$

Observation

We don't want to confuse exponential and power functions. Note that in an **exponential** function

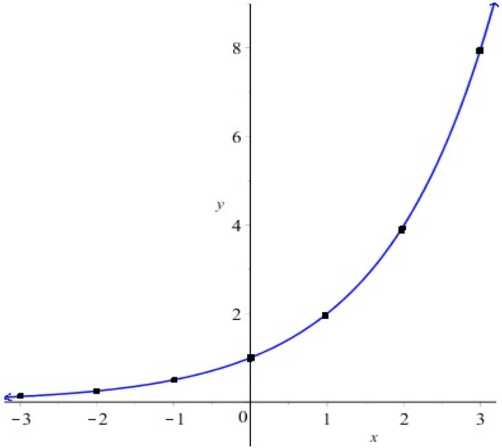
$$f(x) = 4^x$$

the base is a constant, and the exponent is a variable. Contrast a **power** function

$$f(x) = x^4$$

in which the base is variable, and the exponent is a constant.

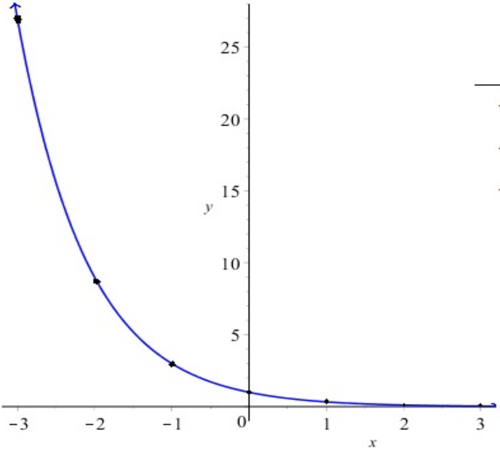
Graphs of Exponential Functions



x	$f(x) = 2^x$
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8

Figure: $f(x) = 2^x$ Note that the function is everywhere increasing. The x-axis is a horizontal asymptote.

Graphs of Exponential Functions



x	$f(x) = (1/3)^x$
-3	27
-2	9
-1	3
0	1
1	$1/3$
2	$1/9$
3	$1/27$

Figure: $f(x) = \left(\frac{1}{3}\right)^x$ Note that the function is everywhere decreasing. The x-axis is a horizontal asymptote.

Graphs of Exponential Functions

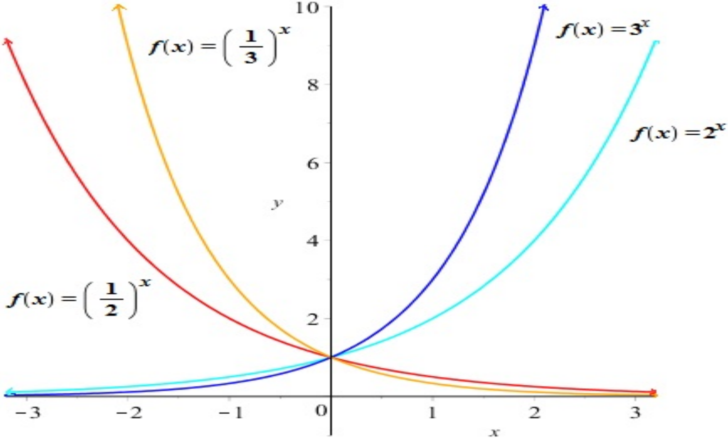


Figure: $f(x) = a^x$ is increasing if $a > 1$ and decreasing if $0 < a < 1$. The line $y = 0$ is a horizontal asymptote for every value of a. Each graph has y-intercept $(0, 1)$. Each graph is strictly above the x-axis.