We considered the **homogeneous** system

\[
\begin{align*}
3x_1 + 5x_2 - 4x_3 &= 0 \\
-3x_1 - 2x_2 + 4x_3 &= 0 \\
6x_1 + x_2 - 8x_3 &= 0
\end{align*}
\]

and found the solutions could be described in **parametric vector form**

\[
x = t \begin{bmatrix} 4 \\ \frac{3}{3} \\ 0 \\ 1 \end{bmatrix}, \quad -\infty < t < \infty
\]

This includes both the **trivial solution** \((x = 0\) when \(t = 0\)) as well as **nontrivial solutions** \((x \neq 0\)).
Nonhomogenous Systems

We then solved the following nonhomogenous system

\[
\begin{align*}
3x_1 &+ 5x_2 - 4x_3 = 7 \\
-3x_1 &- 2x_2 + 4x_3 = -1 \\
6x_1 &+ x_2 - 8x_3 = -4
\end{align*}
\]

and found that the solutions could be described in parametric vector form

\[
\mathbf{x} = \begin{bmatrix}
-1 \\
2 \\
0
\end{bmatrix} + t \begin{bmatrix}
4/3 \\
0 \\
1
\end{bmatrix}, \quad -\infty < t < \infty
\]

We observe that this has the form \( \mathbf{x} = \mathbf{p} + t\mathbf{v} \) where \( \mathbf{v} \) was the solution to the associated homogeneous equation (one with the same left hand side).
Theorem

Suppose the equation $Ax = b$ is consistent for a given $b$. Let $p$ be a solution. Then the solution set of $Ax = b$ is the set of all vectors of the form

$$x = p + vh,$$

where $vh$ is any solution of the associated homogeneous equation $Ax = 0$.

We can use a row reduction technique to get all parts of the solution in one process.
Example

Find the solution set of the following system. Express the solution set in parametric vector form.

\[
\begin{align*}
x_1 + x_2 - 2x_3 + 4x_4 &= 1 \\
2x_1 + 3x_2 - 6x_3 + 12x_4 &= 4
\end{align*}
\]

Has augmented matrix

\[
\begin{bmatrix}
1 & 1 & -2 & 4 & 1 \\
2 & 3 & -6 & 12 & 4
\end{bmatrix}
\]

ref \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 & -1 \\
0 & 1 & -2 & 4 & 2
\end{bmatrix}
\[ x_1 = -1 \]
\[ x_2 = 2 + 2x_3 - 4x_4 \]
\[ x_3, x_4 \text{ - free} \]

To get parametric vector form

\[ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 + 2x_3 - 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2x_3 \\ x_3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \]

[Diagram showing vectors \( \mathbf{p} \) and \( \mathbf{v}_{n} \)]
Section 1.7: Linear Independence

We already know that a homogeneous equation $Ax = 0$ can be thought of as an equation in the column vectors of the matrix $A = [a_1 \ a_2 \ \cdots \ a_n]$ as

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = 0.$$ 

And, we know that at least one solution (the trivial one $x_1 = x_2 = \cdots = x_n = 0$) always exists.

Whether or not there is a nontrivial solution gives us a way to characterize the vectors $a_1, \ldots, a_n$. 
Definition: Linear Dependence/Independence

An indexed set of vectors \( \{ v_1, v_2, \ldots, v_p \} \) in \( \mathbb{R}^n \) is said to be **linearly independent** if the vector equation

\[
x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0
\]

has only the trivial solution.

The set \( \{ v_1, v_2, \ldots, v_p \} \) is said to be **linearly dependent** if there exists a set of weights \( c_1, c_2, \ldots, c_p \) at least one of which is nonzero such that

\[
c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0.
\]

(i.e. Provided the homogeneous equation posses a nontrivial solution.)

An equation \( c_1 v_1 + c_2 v_2 + \cdots c_p v_p = 0 \), with at least one \( c_i \neq 0 \), is called a **linear dependence relation**.
Special Cases

A set with two vectors \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) is linearly dependent if one is a scalar multiple of the other.

Suppose \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) is lin. dependent. Then there exist \( c_1, c_2 \), at least one being nonzero such that

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = \mathbf{0}.
\]

Suppose \( c_1 \neq 0 \).

Then

\[
\mathbf{v}_1 = -\frac{c_2}{c_1} \mathbf{v}_2.
\]

Divide by \( c_1 \)

\[
\mathbf{v}_1 = \frac{-c_2}{c_1} \mathbf{v}_2 = k \mathbf{v}_2 \quad \text{where} \quad k = \frac{-c_2}{c_1}.
\]

So lin. dependence is equivalent to one vector being a multiple of the other.
Example

Determine if the set is linearly dependent or linearly independent.

(a) \( \mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \)

Let \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) be vectors in \( \mathbb{R}^2 \). A linear dependence relation is \( \mathbf{v}_1 + 2\mathbf{v}_2 = \mathbf{0} \).

Note \( \mathbf{v}_1 = -2\mathbf{v}_2 \) (also \( \mathbf{v}_2 = -\frac{1}{2}\mathbf{v}_1 \)).

(b) \( \mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \)

Let \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) be vectors in \( \mathbb{R}^2 \). It is true that \( c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0} \) only if \( c_1 = c_2 = 0 \).
More than Two Vectors

**Theorem:** The columns of a matrix $A$ are linearly independent if and only if the homogeneous equation $Ax = 0$ has only the trivial solution.

$$Ax = 0 \text{ is the vector equation}$$

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n = \vec{0}$$

**where**

$$A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$$
Example

Determine if the set of vectors is linearly dependent or independent. If dependent, find a linear dependence relation.

(a) \[ \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 3 \end{bmatrix} \]

We can create a matrix using the vectors as columns.

Let \( \mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 3 \end{bmatrix} \)

Consider \( \mathbf{A} \mathbf{x} = \mathbf{0} \) by doing rref on \( \begin{bmatrix} \mathbf{A} & \mathbf{0} \end{bmatrix} \)

\[
\begin{bmatrix} \mathbf{A} & \mathbf{0} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]
Solution reads as

\[ x_1 = 0 \]
\[ x_2 = 0 \]
\[ x_3 = 0 \]

No free variables. Hence \( Ax = 0 \) has only the trivial solution.

The vectors are linearly independent.
Determine if the set of vectors is linearly dependent or independent. If dependent, find a linear dependence relation.

\[ \begin{align*}
&\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\} \\
&\text{Let } A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 2 & 3 & 2 \\ 0 & 1 & 3 & 0 \end{bmatrix} \\
&\text{ref of } \begin{bmatrix} A & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&\text{Again we create a matrix. Consider } Ax = 0
\end{align*} \]
We have a free variable, hence non-trivial solutions to \( Ax = 0 \). The vectors are linearly dependent.

We can interpret the rref as

\[
\vec{v}_4 = \frac{1}{3} \vec{v}_1 + 2 \vec{v}_2 - \frac{2}{3} \vec{v}_3.
\]

A linear dependence relation is

\[
-\frac{1}{3} \vec{v}_1 - 2 \vec{v}_2 + \frac{2}{3} \vec{v}_3 + \vec{v}_4 = \vec{0}
\]
Theorem

An indexed set of two or more vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others in the set.

Example: Let \( \mathbf{u} \) and \( \mathbf{v} \) be any nonzero vectors in \( \mathbb{R}^3 \). Show that if \( \mathbf{w} \) is any vector in \( \text{Span}\{\mathbf{u}, \mathbf{v}\} \), then the set \( \{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \) is linearly dependent.

Since \( \mathbf{w} \) is in \( \text{Span}\{\mathbf{u}, \mathbf{v}\} \), then exist scalars \( c_1, c_2 \) such that

\[
\mathbf{w} = c_1 \mathbf{u} + c_2 \mathbf{v}.
\]

Subtract \( \mathbf{w} \) to get
\[ c_1 \vec{u} + c_2 \vec{v} - \vec{w} = \vec{0} \]

The coefficients are \( c_1, c_2 \), and \(-1\). At least one of these is non-zero (since \(-1 \neq 0\)).

This is a linear dependence relation. 
\( \{ \vec{u}, \vec{v}, \vec{w} \} \) is linearly dependent.
Caveat!

A set may be linearly dependent even if all proper subsets are linearly independent. For example, consider

\[
\begin{bmatrix}
1 \\
0 \\
0 
\end{bmatrix}
, \quad
\begin{bmatrix}
1 \\
1 \\
0 
\end{bmatrix}
, \quad\text{and}\quad
\begin{bmatrix}
0 \\
0 \\
1 
\end{bmatrix}
.
\]

Examine each set \(\{\mathbf{v}_1, \mathbf{v}_2\}\), \(\{\mathbf{v}_1, \mathbf{v}_3\}\), \(\{\mathbf{v}_2, \mathbf{v}_3\}\), and \(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\).

The first 3 sets are lin independent.

But

\[
\mathbf{v}_2 = \mathbf{v}_2 - \mathbf{v}_1
\]

so, \(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\) is lin dependent.
Two More Theorems

**Theorem:** If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, if \( \{ v_1, v_2, \ldots, v_p \} \) is a set of vectors in \( \mathbb{R}^n \), and \( p > n \), then the set is linearly dependent.

- e.g. 5 vectors from \( \mathbb{R}^3 \)
- 16 vectors from \( \mathbb{R}^n \)

**Theorem:** Any set of vectors that contains the zero vector is linearly dependent.
Determine if the set is linearly dependent or linearly independent

(a) \[
\begin{bmatrix}
1 & 2 \\
2 & -1
\end{bmatrix},
\begin{bmatrix}
3 & 3 \\
3 & -5
\end{bmatrix},
\begin{bmatrix}
0 & 1 \\
1 & -1
\end{bmatrix},
\begin{bmatrix}
2 & 3 \\
3 & 3
\end{bmatrix}
\] 

4 vectors in \( \mathbb{R}^3 \)

Lin. dependent.
Determine if the set is linearly dependent or linearly independent

(b) \[ \left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -8 \end{bmatrix}, \begin{bmatrix} \end{bmatrix} \right\} \]

Lin. Dependent, contains 0.
Determine if the set is linearly dependent or linearly independent

(c) \[ \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 2 \\ 2 \end{bmatrix} \right\} \]

Note \( \vec{v}_2 = -2 \vec{v}_1 \)

Lin. Dependent.