February 1 Math 3260 sec. 55 Spring 2018

Section 1.5: Solution Sets of Linear Systems

We considered the **homogeneous** system

$$\begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

and found the solutions could be described in **parametric vector form**

$$\mathbf{x} = t\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}, \quad -\infty < t < \infty$$

This includes both the **trivial solution** ($\mathbf{x} = \mathbf{0}$ when $t = 0$) as well as **nontrivial solutions** ($\mathbf{x} \neq \mathbf{0}$).

Nonhomogenous Systems

We then solved the following **nonhomogenous** system

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 7 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & -1 \\ 6x_1 & + & x_2 & - & 8x_3 & = & -4 \end{array}$$

and found that the solutions could be described in parametric vector form

$$\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}, \quad -\infty < t < \infty$$

We observe that this has the form $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ where $\mathbf{v}$ was the solution to the associated homogeneous equation (one with the same left hand side).

Theorem

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for a given $\mathbf{b}$. Let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{x} = \mathbf{p} + \mathbf{v}_h,$$

where $\mathbf{v}_h$ is any solution of the associated homogeneous equation $A\mathbf{x} = \mathbf{0}$.

We can use a row reduction technique to get all parts of the solution in one process.

Example

Find the solution set of the following system. Express the solution set in parametric vector form.

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & 2x_3 & + & 4x_4 & = & 1 \\ 2x_1 & + & 3x_2 & - & 6x_3 & + & 12x_4 & = & 4 \end{array}$$

Has augmented matrix

$$\begin{bmatrix} 1 & 1 & -2 & 4 & 1 \\ 2 & 3 & -6 & 12 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 4 & 2 \end{bmatrix}$$

$$x_1 = -1$$

$$x_2 = 2 + 2x_3 - 4x_4$$

$$x_3, x_4 \text{ - free}$$

To get parametric vector form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 + 2x_3 - 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -4x_4 \\ 0 \\ x_4 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}}_{\vec{p}} + \underbrace{x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_h}$$

Section 1.7: Linear Independence

We already know that a homogeneous equation $A\mathbf{x} = \mathbf{0}$ can be thought of as an equation in the column vectors of the matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n = \mathbf{0}.$$

And, we know that at least one solution (the trivial one $x_1 = x_2 = \cdots = x_n = 0$) always exists.

Whether or not there is a nontrivial solution gives us a way to characterize the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Definition: Linear Dependence/Independence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists a set of weights $c_1, c_2, \dots, c_p$ *at least one of which is nonzero* such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}.$$

(i.e. Provided the homogeneous equation posses a nontrivial solution.)

An equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}$, with at least one $c_i \neq 0$, is called a **linear dependence relation**.

Special Cases

A set with two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if one is a scalar multiple of the other.

Suppose $\{\vec{v}_1, \vec{v}_2\}$ is lin. dependent. Then there exist c_1, c_2, at least one being nonzero such that

$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. Suppose $c_1 \neq 0$.

Then $c_1\vec{v}_1 = -c_2\vec{v}_2$. Divide by c_1

$$\vec{v}_1 = \frac{-c_2}{c_1}\vec{v}_2 = k\vec{v}_2 \quad \text{where } k = \frac{-c_2}{c_1}.$$

So lin. dependence is equivalent to one vector being a multiple of the other.

Example

Determine if the set is linearly dependent or linearly independent.

(a) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

Lin. dependent

note $\vec{v}_1 = -2\vec{v}_2$ (also $\vec{v}_2 = \frac{-1}{2}\vec{v}_1$)

A lin. dependence relation is $\vec{v}_1 + 2\vec{v}_2 = \vec{0}$.

(b) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Lin. independent

$\vec{v}_1 \neq k\vec{v}_2$ for any one number k.

It is true that

$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ only if $c_1 = c_2 = 0$

More than Two Vectors

Theorem: The columns of a matrix A are linearly **independent** if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

$A\vec{x} = \vec{0}$ is the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{0}$$

where $A = [\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_n]$

Example

Determine if the set of vectors is linearly dependent or independent. If dependent, find a linear dependence relation.

(a) $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix} \right\}$

we can create a matrix using the vectors as columns.

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 3 \end{bmatrix}$

Consider $A\vec{x} = \vec{0}$ by doing rref on $[A \ \vec{0}]$

$$[A \ \vec{0}] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution reads as

$$x_1 = 0$$
$$x_2 = 0$$
$$x_3 = 0$$

No free variables. Hence $A\vec{x} = \vec{0}$ has only the trivial solution.

The vectors are linearly independent.

Determine if the set of vectors is linearly dependent or independent. If dependent, find a linear dependence relation.

(b) $$\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}}, \underset{\vec{v}_4}{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}} \right\}$$

Again we create a matrix.

Consider $A\vec{x} = \vec{0}$

Let $A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 2 & 3 & 2 \\ 0 & 1 & 3 & 0 \end{bmatrix}$

rref of $\begin{bmatrix} A & \vec{0} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We have a free variable, hence nontrivial solutions to $A\vec{x} = \vec{0}$. The vectors are linearly dependent.

We can interpret the rref as

$$\vec{v}_4 = \frac{1}{3}\vec{v}_1 + 2\vec{v}_2 - \frac{2}{3}\vec{v}_3 .$$

A linear dependence relation is

$$-\frac{1}{3}\vec{v}_1 - 2\vec{v}_2 + \frac{2}{3}\vec{v}_3 + \vec{v}_4 = \vec{0}$$

Theorem

An indexed set of two or more vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others in the set.

Example: Let **u** and **v** be any nonzero vectors in $\mathbb{R}^3$. Show that if **w** is any vector in Span$\{\mathbf{u}, \mathbf{v}\}$, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly **dependent.**

Since $\vec{w}$ is in Span$\{\vec{u}, \vec{v}\}$, then exist scalers c_1, c_2 such that

$$\vec{w} = c_1\vec{u} + c_2\vec{v}.$$

Subtract $\vec{w}$ to get

$$c_1\vec{u} + c_2\vec{v} - \vec{w} = \vec{0}$$

The coefficients are c_1, c_2, and -1. At least one of these is nonzero (since $-1 \neq 0$).

This is a linear dependence relation.

$\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent.

Caveat!

A set may be linearly dependent even if all proper subsets are linearly independent. For example, consider

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Examine each set $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, $\{\mathbf{v}_2, \mathbf{v}_3\}$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

The first 3 sets are lin independent.

But

$$\vec{v}_3 = \vec{v}_2 - \vec{v}_1$$

so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is lin. dependent.

Two More Theorems

Theorem: If a set contains more vectors than there are entries in each vector, then the set is linearly **dependent**. That is, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a set of vector in $\mathbb{R}^n$, and $p > n$, then the set is linearly dependent.

e.g. 5 vectors from $\mathbb{R}^3$

16 vectors from $\mathbb{R}^{11}$

Theorem: Any set of vectors that contains the zero vector is linearly **dependent**.

Determine if the set is linearly dependent or linearly independent

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$

4 vectors in $\mathbb{R}^3$

Lin. dependent.

Determine if the set is linearly dependent or linearly independent

(b) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -8 \\ 1 \end{bmatrix}, \right\}$

Lin. Dependent, contains $\vec{0}$.

Determine if the set is linearly dependent or linearly independent

(c) $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 2 \\ 2 \end{bmatrix} \right\}$

$\vec{v}_1 \qquad \vec{v}_2$

Note $\vec{v}_2 = -2\vec{v}_1$

Lin. Dependent.