February 1 Math 3260 sec. 56 Spring 2018

Section 1.5: Solution Sets of Linear Systems

We considered the **homogeneous** system

$$\begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

and found the solutions could be described in **parametric vector form**

$$\mathbf{x} = t\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}, \quad -\infty < t < \infty$$

This includes both the **trivial solution** ($\mathbf{x} = \mathbf{0}$ when $t = 0$) as well as **nontrivial solutions** ($\mathbf{x} \neq \mathbf{0}$).

Nonhomogenous Systems

We then solved the following **nonhomogenous** system

$$\begin{array}{rcrcrcr}
3x_1 & + & 5x_2 & - & 4x_3 & = & 7\\
-3x_1 & - & 2x_2 & + & 4x_3 & = & -1\\
6x_1 & + & x_2 & - & 8x_3 & = & -4
\end{array}$$

and found that the solutions could be described in parametric vector form

$$\mathbf{x} = \begin{bmatrix} -1\\ 2\\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{4}{3}\\ 0\\ 1 \end{bmatrix}, \quad -\infty < t < \infty$$

We observe that this has the form $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ where $\mathbf{v}$ was the solution to the associated homogeneous equation (one with the same left hand side).

Theorem

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for a given $\mathbf{b}$. Let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{x} = \mathbf{p} + \mathbf{v}_h,$$

where $\mathbf{v}_h$ is any solution of the associated homogeneous equation $A\mathbf{x} = \mathbf{0}$.

We can use a row reduction technique to get all parts of the solution in one process.

Example

Find the solution set of the following system. Express the solution set in parametric vector form.

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & - & 2x_3 & + & 4x_4 & = & 1 \\ 2x_1 & + & 3x_2 & - & 6x_3 & + & 12x_4 & = & 4 \end{array}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & -2 & 4 & 1 \\ 2 & 3 & -6 & 12 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 4 & 2 \end{bmatrix}$$

From the rref

$$x_1 = -1$$
$$x_2 = 2 + 2x_3 - 4x_4$$
$$x_3, x_4 - \text{free}$$

The solutions $\vec{x}$ have the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 + 2x_3 - 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -4x_4 \\ 0 \\ x_4 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}}_{\vec{p}} + \underbrace{x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_h}$$

Section 1.7: Linear Independence

We already know that a homogeneous equation $A\mathbf{x} = \mathbf{0}$ can be thought of as an equation in the column vectors of the matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n = \mathbf{0}.$$

And, we know that at least one solution (the trivial one $x_1 = x_2 = \cdots = x_n = 0$) always exists.

Whether or not there is a nontrivial solution gives us a way to characterize the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Definition: Linear Dependence/Independence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists a set of weights $c_1, c_2, \dots, c_p$ *at least one of which is nonzero* such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}.$$

(i.e. Provided the homogeneous equation posses a nontrivial solution.)

An equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}$, with at least one $c_i \neq 0$, is called a **linear dependence relation**.

Special Cases

A set with two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if one is a scalar multiple of the other.

Suppose $\{\vec{v}_1, \vec{v}_2\}$ is lin. dependent. Then there exists c_1, c_2 with at least one nonzero such that

$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. Assume $c_1 \neq 0$. Then

$c_1\vec{v}_1 = -c_2\vec{v}_2$. Divide by c_1

$\vec{v}_1 = \frac{-c_2}{c_1}\vec{v}_2 = k\vec{v}_2$ where $k = \frac{-c_2}{c_1}$.

So for 2 vectors, linear dependence is equivalent to one being a multiple of the other.

Example

Determine if the set is linearly dependent or linearly independent.

(a) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

Lin. Dependent.

$\vec{v}_1 = -2\vec{v}_2$ i.e. $\vec{v}_2 = \frac{-1}{2}\vec{v}_1$

A linear dependence relation is $\vec{v}_1 + 2\vec{v}_2 = \vec{0}$

(b) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Lin. Independent

Note $\vec{v}_1 \neq k\vec{v}_2$ for any one scaler k.

More than Two Vectors

Theorem: The columns of a matrix A are linearly **independent** if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

$A\vec{x} = \vec{0}$ is equivalent to the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{0}$$

If $A = [\vec{a}_1 \; \vec{a}_2 \; \cdots \; \vec{a}_n]$.

Example

Determine if the set of vectors is linearly dependent or independent. If dependent, find a linear dependence relation.

(a) $\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}} \right\}$

Will use a matrix and consider an equation $A\vec{x} = \vec{0}$.

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 3 \end{bmatrix}$

$[A \ \vec{0}] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

no free variables

$\Rightarrow$ no nontrivial solutions

$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0 \quad$ i.e. $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$.

Since $A\vec{x} = \vec{0}$ has only the trivial solution, the set of vectors is linearly independent.

Determine if the set of vectors is linearly dependent or independent. If dependent, find a linear dependence relation.

(b) $\left\{ \underbrace{\begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}}_{\vec{v}_3}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}}_{\vec{v}_4} \right\}$

Again we'll consider an equation

$A\vec{x} = \vec{0}$

Let $A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 2 & 3 & 2 \\ 0 & 1 & 3 & 0 \end{bmatrix}$

$[A \ \vec{0}] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

free variable

Since there's a free variable, there are nontrivial solutions to $A\vec{x} = \vec{0}$.

The set is linearly dependent.

From the matrix

$$\vec{v}_4 = \frac{1}{3}\vec{v}_1 + 2\vec{v}_2 - \frac{2}{3}\vec{v}_3$$

A linear dependence relation is

$$-\frac{1}{3}\vec{v}_1 - 2\vec{v}_2 + \frac{2}{3}\vec{v}_3 + \vec{v}_4 = \vec{0}$$

Theorem

An indexed set of two or more vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others in the set.

Example: Let **u** and **v** be any nonzero vectors in $\mathbb{R}^3$. Show that if **w** is any vector in Span$\{\mathbf{u}, \mathbf{v}\}$, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly **dependent.**

Suppose $\vec{w}$ is in Span$\{\vec{u}, \vec{v}\}$. Then there exist scalers c_1, c_2 such that

$$\vec{w} = c_1\vec{u} + c_2\vec{v}.$$

From this, $c_1\vec{u} + c_2\vec{v} - \vec{w} = \vec{0}$.

This is a linear dependence relation on $\{\vec{u}, \vec{v}, \vec{w}\}$ because at least one of the coefficients is nonzero (since $-1 \neq 0$).

The set must be linearly dependent.

Caveat!

A set may be linearly dependent even if all proper subsets are linearly independent. For example, consider

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Examine each set $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, $\{\mathbf{v}_2, \mathbf{v}_3\}$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

The sets of 2 are all linearly independent. But $\vec{v}_3 = \vec{v}_2 - \vec{v}_1$.

Hence the set of all three is linearly dependent.

Two More Theorems

Theorem: If a set contains more vectors than there are entries in each vector, then the set is linearly **dependent**. That is, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a set of vector in $\mathbb{R}^n$, and $p > n$, then the set is linearly dependent.

E.g. 5 vectors in $\mathbb{R}^3$

or 26 vectors in $\mathbb{R}^{17}$

Theorem: Any set of vectors that contains the zero vector is linearly **dependent**.

Determine if the set is linearly dependent or linearly independent

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$

4 vectors in $\mathbb{R}^3$, they're linearly dependent.

Determine if the set is linearly dependent or linearly independent

(b) $\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}}, \underset{\vec{0}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 2 \\ 4 \\ -8 \\ 1 \end{bmatrix}}, \right\}$

Note

$0\vec{v}_1 + 37\vec{0} + 0\vec{v}_2 = \vec{0}$

a lin. dependence relation

Contains the zero vector $\Rightarrow$

lin. dependent

Determine if the set is linearly dependent or linearly independent

(c) $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 2 \\ 2 \end{bmatrix} \right\}$

$\vec{v}_1 \qquad \vec{v}_2$

$\vec{v}_2 = -2\vec{v}_1$

One's a multiple of the other.

They are lin. dependent.