February 20 Math 2306 sec. 54 Spring 2019

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

We found that $y=e^{mx}$ is a solution provided m is a solution to the equation

$$am^2+bm+c=0$$

called the characteristic (or auxiliary) equation.

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

The values of m are

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Example

2^{nd} order, linear, homogeneous, constant coefficient

Solve the IVP

$$y'' + y' - 12y = 0, \quad y(0) = 1, \quad y'(0) = 10$$

Characteristic equation: $m^2 + m - 12 = 0$

$$(m+4)(m-3) = 0$$

$$m_1 = -4, \quad m_2 = 3$$

Solutions $y_1 = e^{-4x}$, $y_2 = e^{3x}$

General solution $y = c_1 e^{-4x} + c_2 e^{3x}$

Impose the conditions $y(0) = 1$, $y'(0) = 10$

$$y' = -4c_1 e^{-4x} + 3c_2 e^{3x}$$

$$y(0) = c_1 e^0 + c_2 e^0 = 1 \quad \Longrightarrow \quad c_1 + c_2 = 1$$

$$y'(0) = -4c_1 e^0 + 3c_2 e^0 = 10 \quad \Longrightarrow \quad -4c_1 + 3c_2 = 10$$

$$\left.\begin{aligned} 4c_1 + 4c_2 &= 4 \\ -4c_1 + 3c_2 &= 10 \end{aligned}\right\} \overset{\text{add}}{\Longrightarrow} \quad 7c_2 = 14 \Rightarrow c_2 = 2$$

$$c_1 = 1 - c_2 = 1 - 2 = -1$$

The solution to the IVP is

$$y = -e^{-4x} + 2e^{3x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$
$$y = c_1 e^{mx} + c_2 x e^{mx} \quad \text{where} \quad m = \frac{-b}{2a}$$

Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$.

Standard form $\quad y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad P(x) = \frac{b}{a}$

$y_2 = uy_1$ where $\quad u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2}\,dx$

$$e^{-\int P(x)dx} = e^{-\int \frac{b}{a}dx} = e^{-\frac{b}{a}x}$$

$$(y_1)^2 = \left(e^{\frac{-b}{2a}x}\right)^2 = e^{2\left(\frac{-b}{2a}x\right)} = e^{\frac{-b}{a}x}$$

$$u = \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx = \int \frac{e^{\frac{-b}{a}x}}{e^{-b/a\,x}}\,dx = \int dx = x$$

$$y_2 = u y_1 = x\,e^{\frac{-b}{2a}x}$$

You can confirm that $y_1 = e^{\frac{-b}{2a}x}$ and $y_2 = x\,e^{\frac{-b}{2a}x}$ are linearly independent.

Example

Solve the ODE

$$4y'' - 4y' + y = 0$$

Characteristic equation: $4m^2 - 4m + 1 = 0$

$$(2m-1)^2 = 0$$

$$m = \frac{1}{2} \quad \text{repeated}$$

Solutions are $y_1 = e^{\frac{1}{2}x}$ and $y_2 = x e^{\frac{1}{2}x}$

The general solution is

$$\boxed{y = C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}}$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad \text{where the roots}$$

$$m = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

The solutions can be written as

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}.$$

$e^{\alpha x + i\beta x}$

Deriving the solutions Case III

$\cos(-\beta x) = \cos(\beta x)$

$\sin(-\beta x) = -\sin(\beta x)$

Recall Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$y_1 = e^{\alpha x} e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$

$$y_2 = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$

$$y_1 = e^{\alpha x}\cos(\beta x) + i e^{\alpha x}\sin(\beta x)$$

$$y_2 = e^{\alpha x}\cos(\beta x) - i e^{\alpha x}\sin(\beta x)$$

By superposition, we can take $y_1 = k_1 \psi_1 + k_2 \psi_2$ for any constants k_1, k_2.

Let $y_1 = \frac{1}{2}(\psi_1 + \psi_2) = \frac{1}{2}\left(2e^{\alpha x}\cos(\beta x) + i \cdot 0\right) = e^{\alpha x}\cos(\beta x)$

$y_2 = \frac{1}{2i}(\psi_1 - \psi_2) = \frac{1}{2i}\left(0 + 2i\, e^{\alpha x}\sin(\beta x)\right) = e^{\alpha x}\sin(\beta x)$

The fundamental solution set is

$$y_1 = e^{\alpha x}\cos(\beta x), \quad y_2 = e^{\alpha x}\sin(\beta x)$$