

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order¹, linear, homogeneous equation with constant coefficients

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0.$$

Question: What sort of function y could be expected to satisfy

$$y'' = \text{constant } y' + \text{constant } y?$$

¹We'll generalize to higher order later in this section.

We look for solutions of the form $y = e^{mx}$ with m constant.

Supposed to solve

$$ay'' + by' + cy = 0$$

Substitute

$$y = e^{mx}, \quad y' = me^{mx}, \quad y'' = m^2 e^{mx}$$

$$am^2 e^{mx} + bm e^{mx} + ce^{mx} = 0$$

$$e^{mx} (am^2 + bm + c) = 0$$

This holds for all x in some interval

If $am^2 + bm + c = 0$

So $y = e^{mx}$ will be a solution if

m is a root of the quadratic

equation $am^2 + bm + c = 0$

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

- I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$
- II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$
- III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates
 $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent.

Here $m_1 \neq m_2$. Using the Wronskian

$$\begin{aligned} W(y_1, y_2)(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix} \end{aligned}$$

$$= e^{m_1 x} \left(m_2 e^{m_2 x} \right) - m_1 e^{m_1 x} \cdot e^{m_2 x}$$

$$= m_2 e^{(m_1+m_2)x} - m_1 e^{(m_1+m_2)x}$$

$$W(y_1, y_2)(x) = (m_2 - m_1) e^{(m_1+m_2)x}$$

Note $W \neq 0$ since $m_2 \neq m_1$.

Hence $y_1 = e^{m_1 x}$, $y_2 = e^{m_2 x}$ give a fundamental

solution set.

Example

Solve the IVP

$$y'' + y' - 12y = 0, \quad y(0) = 1, \quad y'(0) = 10$$

Gen. soln. Characteristic eqn.

$$\begin{aligned} m^2 + m - 12 &= 0 \\ (m+4)(m-3) &= 0 \Rightarrow m_1 = -4, \quad m_2 = 3 \end{aligned}$$

$y_1 = e^{-4x}$, $y_2 = e^{3x}$ the general solution

$$y = C_1 e^{-4x} + C_2 e^{3x}$$

$$y' = -4C_1 e^{-4x} + 3C_2 e^{3x}$$

$$y(0) = C_1 e^0 + C_2 e^0 = 1 \Rightarrow C_1 + C_2 = 1$$

$$y'(0) = -4C_1 e^0 + 3C_2 e^0 = 10 \Rightarrow -4C_1 + 3C_2 = 10$$

$$4C_1 + 4C_2 = 4$$

add

$$7C_2 = 14$$

$$C_2 = 2$$

$$C_1 = 1 - C_2 = 1 - 2 = -1$$

The solution to the IVP is

$$y = -e^{-4x} + 2e^{3x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

$$y = c_1 e^{mx} + c_2 x e^{mx} \quad \text{where} \quad m = \frac{-b}{2a}$$

Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$.

Standard form:

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad y_1 = e^{\frac{-bx}{2a}} \text{ known.}$$

$$y_2 = uy_1, \quad \text{where} \quad u = \int \frac{e^{-\int p(x)dx}}{(y_1)^2} dx$$

$$P(x) = \frac{b}{a}, \quad -\int P(x) dx = -\int \frac{b}{a} dx = -\frac{b}{a} x$$

$$u = \int \frac{-\frac{b}{a}x}{\left(\frac{e^{-\frac{b}{2a}x}}{e}\right)^2} dx = \int \frac{-\frac{b}{a}x}{e^{-2\left(\frac{b}{2a}x\right)}} dx$$

$$\therefore \int \frac{e^{\frac{-b}{a}x}}{e^{-\frac{b}{a}x}} dx = \int dx = x$$

$$y_2 = u y_1 = x e^{\frac{-b}{2a}x}$$

Example

Solve the ODE

$$4y'' - 4y' + y = 0$$

Characteristic eqn $4m^2 - 4m + 1 = 0$

$$(2m-1)^2 = 0 \Rightarrow m = \frac{1}{2}$$

repeated root

$$y_1 = e^{\frac{1}{2}x}, \quad y_2 = x e^{\frac{1}{2}x}$$

General solution $\underline{\underline{y = C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}}}$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad \text{where the roots}$$

$$m = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

α, β are real numbers we'll always take $\beta > 0$.

The solutions can be written as

$$Y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x} e^{-i\beta x}.$$

Deriving the solutions Case III

Recall Euler's Formula:

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \theta - \text{real valued}$$

$$\begin{aligned} Y_1 &= e^{\alpha x} e^{i\beta x} = e^{\alpha x} (\cos(\beta x) + i \sin(\beta x)) \\ &= e^{\alpha x} \cos(\beta x) + i e^{\alpha x} \sin(\beta x) \end{aligned}$$

$$\begin{aligned} Y_2 &= e^{\alpha x} e^{-i\beta x} = e^{\alpha x} (\cos(\beta x) - i \sin(\beta x)) \\ &= e^{\alpha x} \cos(\beta x) - i e^{\alpha x} \sin(\beta x) \end{aligned}$$

Using the principle of superposition, we'll take

$$y_1 = \frac{1}{2} (Y_1 + Y_2) \text{ and } y_2 = \frac{1}{2i} (Y_1 - Y_2)$$

$$y_1 = \frac{1}{2} \left(2 e^{\alpha x} \cos(\beta x) + 0 \right) = e^{\alpha x} \cos(\beta x)$$

$$y_2 = \frac{1}{2i} \left(0 + 2i e^{\alpha x} \sin(\beta x) \right) = e^{\alpha x} \sin(\beta x)$$

Our fundamental solution set is

$$y_1 = e^{\alpha x} \cos(\beta x), \quad y_2 = e^{\alpha x} \sin(\beta x)$$

Example

Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0$$

Char. eqn $m^2 + 4m + 6 = 0$

$$m = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 6}}{2 \cdot 1} = \frac{-4 \pm 2\sqrt{2}i}{2}$$

$$= -2 \pm \sqrt{2}i$$

$$x_1 = e^{-2t} \cos(\sqrt{2}t), x_2 = e^{-2t} \sin(\sqrt{2}t)$$

$$\alpha = -2, \beta = \sqrt{2}$$

Gen. soln

$$x = c_1 e^{-2t} \cos(\sqrt{2}t) + c_2 e^{-2t} \sin(\sqrt{2}t)$$

Higer Order Linear Constant Coefficient ODEs

- ▶ The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.
- ▶ Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x} \cos(\beta x)$ and $e^{\alpha x} \sin(\beta x)$.
- ▶ If a root m is repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2 e^{mx}, \quad \dots, \quad x^{k-1} e^{mx}$$

or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x} \cos(\beta x), \quad e^{\alpha x} \sin(\beta x), \quad xe^{\alpha x} \cos(\beta x), \quad xe^{\alpha x} \sin(\beta x), \quad \dots,$$

$$x^{k-1} e^{\alpha x} \cos(\beta x), \quad x^{k-1} e^{\alpha x} \sin(\beta x)$$

- ▶ It may require a computer algebra system to find the roots for a high degree polynomial.

Example

Solve the ODE

$$y''' - 4y' = 0$$

Substitute $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2e^{mx}$, $y''' = m^3e^{mx}$

$$m^3e^{mx} - 4me^{mx} = 0$$
$$e^{mx}(m^3 - 4m) = 0$$

The characteristic eqn is

$$m^3 - 4m = 0$$

factor $m(m^2 - 4) = 0$ $m_1 = 0$

$$m(m-2)(m+2) = 0 \Rightarrow m_2 = 2$$

$$m_3 = -2$$

3 soln.

$$y_1 = e^{0x} = 1, \quad y_2 = e^{2x}, \quad y_3 = e^{-2x}$$

Gen. soln

$$y = C_1 + C_2 e^{2x} + C_3 e^{-2x}$$