February 20 Math 2306 sec. 60 Spring 2018

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order[1], linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

Question: What sort of function y could be expected to satisfy

$$y''=\text{constant}\,y'+\text{constant}\,y?$$

[1]We'll generalize to higher order later in this section.

We look for solutions of the form $y = e^{mx}$ with m constant.

y is to solve $ay'' + by' + cy = 0$

Substitute

$$y = e^{mx}, \quad y' = me^{mx}, \quad y'' = m^2 e^{mx}$$

$$ay'' + by' + cy = am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$e^{mx}\left(am^2 + bm + c\right) = 0$$

This will hold if m is a root of the quadratic equation

$$am^2 + bm + c = 0$$

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent.

Using the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$$

$$= e^{m_1 x}\left(m_2 e^{m_2 x}\right) - m_1 e^{m_1 x}\left(e^{m_2 x}\right)$$

$$= m_2 e^{(m_1+m_2)x} - m_1 e^{(m_1+m_2)x}$$

$$W(y_1, y_2)(x) = (m_2 - m_1) e^{(m_1+m_2)x}$$

Since $W \neq 0$, $m_2 \neq m_1$

y_1 and y_2 are linearly independent.

Example

Solve the IVP

$$y'' + y' - 12y = 0, \quad y(0) = 1, \quad y'(0) = 10$$

Find general solution:

Characteristic eqn

$$m^2 + m - 12 = 0$$

$$(m+4)(m-3) = 0 \quad \Longrightarrow \quad m_1 = -4, \ m_2 = 3$$

$$y_1 = e^{-4x}, \quad y_2 = e^{3x}$$

Gen. soln. $\quad y = c_1 e^{-4x} + c_2 e^{3x}$

$$y' = -4c_1 e^{-4x} + 3c_2 e^{3x}$$

Apply $y(0)=1$, $y'(0)=10$

$$y(0) = c_1 e^0 + c_2 e^0 = 1 \Rightarrow \quad c_1 + c_2 = 1$$

$$y'(0) = -4c_1 e^0 + 3c_2 e^0 = 10 \Rightarrow \quad -4c_1 + 3c_2 = 10$$

$$4c_1 + 4c_2 = 4$$

add

$$7c_2 = 14$$

$$c_2 = 2$$

$$c_1 = 1 - c_2 = -1$$

The solution to the IVP is

$$\boxed{y = -e^{-4x} + 2e^{3x}}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$
$$y = c_1 e^{mx} + c_2 x e^{mx} \quad \text{where} \quad m = \frac{-b}{2a}$$

Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$.

Standard form

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \qquad y_1 = e^{-\frac{b}{2a}x}$$

$$y_2 = uy_1 \quad \text{where} \quad u = \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx$$

$P(x) = \frac{b}{a}$ $\quad e^{-\int P(x)dx} = e^{-\int \frac{b}{a} dx} = e^{-\frac{b}{a}x}$

$$u = \int \frac{e^{-\frac{b}{a}x}}{\left(e^{\frac{-b}{2a}x}\right)^2} dx = \int \frac{e^{-\frac{b}{a}x}}{e^{2\left(\frac{-b}{2a}x\right)}} dx$$

$$= \int \frac{e^{\frac{-b}{a}x}}{e^{\frac{-b}{a}x}} dx = \int dx = x$$

So $\quad y_2 = xe^{\frac{-b}{2a}x}$

Example

Solve the ODE

$$4y'' - 4y' + y = 0$$

Characteristic eqn : $4m^2 - 4m + 1 = 0$

$(2m-1)^2 = 0 \Rightarrow m = \frac{1}{2}$ repeated

$y_1 = e^{\frac{1}{2}x}$, $y_2 = x e^{\frac{1}{2}x}$

The general solution is $y = C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad \text{where the roots}$$

$$m = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

α, β are real $\quad \beta > 0$.

The solutions can be written as

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}.$$

Deriving the solutions Case III

Recall Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

θ - real

$$y_1 = e^{\alpha x} e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$

$$= e^{\alpha x}\cos(\beta x) + i e^{\alpha x}\sin(\beta x)$$

$$y_2 = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$

$$= e^{\alpha x}\cos(\beta x) - i e^{\alpha x}\sin(\beta x)$$

Using superposition, we will take

$$y_1 = \frac{1}{2}(\varphi_1 + \varphi_2) \qquad y_2 = \frac{1}{2i}(\varphi_1 - \varphi_2)$$

$$y_1 = \frac{1}{2}(\varphi_1 + \varphi_2) = \frac{1}{2}\left(2e^{\alpha x}\cos(\beta x) + 0\right) = e^{\alpha x}\cos(\beta x)$$

$$y_2 = \frac{1}{2i}(\varphi_1 - \varphi_2) = \frac{1}{2i}\left(0 + 2i\,e^{\alpha x}\sin(\beta x)\right) = e^{\alpha x}\sin(\beta x)$$

A fundamental sln. set is

$$y_1 = e^{\alpha x}\cos(\beta x), \quad y_2 = e^{\alpha x}\sin(\beta x).$$

Example

Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0$$

Char. eqn $m^2 + 4m + 6 = 0$

$$m = \frac{-4 \pm \sqrt{4^2 - 4\cdot1\cdot6}}{2\cdot1} = \frac{-4 \pm \sqrt{-8}}{2}$$

$$= \frac{-4 \pm 2\sqrt{2}\,i}{2} = -2 \pm \sqrt{2}\,i$$

$$\alpha = -2, \quad \beta = \sqrt{2}$$

$$x_1 = e^{-2t}\cos(\sqrt{2}\,t), \quad x_2 = e^{-2t}\sin(\sqrt{2}\,t)$$

General solution $x = c_1 e^{-2t}\cos(\sqrt{2}\,t) + c_2 e^{-2t}\sin(\sqrt{2}\,t)$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$.
- If a root m is repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2e^{mx}, \quad \ldots, \quad x^{k-1}e^{mx}$$

or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \ldots,$$

$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

- It may require a computer algebra system to find the roots for a high degree polynomial.

Example

Solve the ODE

$y''' - 4y' = 0$

If $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$y''' = m^3 e^{mx}$

$$y''' - 4y' = m^3 e^{mx} - 4me^{mx} = 0$$

$$e^{mx}(m^3 - 4m) = 0$$

Char. eqn is

$$m^3 - 4m = 0$$

$$m(m^2 - 4) = 0$$

$$m(m+2)(m-2) = 0$$

$m_1 = 0$, $m_2 = -2$, $m_3 = 2$

$y_1 = e^{0x} = 1$, $y_2 = e^{-2x}$, $y_3 = e^{2x}$

The general soln.

$$\boxed{y = C_1 + C_2 e^{-2x} + C_3 e^{2x}.}$$