

## Section 2.2: Inverse of a Matrix

Consider the scalar equation  $ax = b$ . Provided  $a \neq 0$ , we can solve this explicitly

$$x = a^{-1}b$$

where  $a^{-1}$  is the unique number such that  $aa^{-1} = a^{-1}a = 1$ .

If  $A$  is an  $n \times n$  matrix, we seek an analog  $A^{-1}$  that satisfies the condition

$$A^{-1}A = AA^{-1} = I_n.$$

If such matrix  $A^{-1}$  exists, we'll say that  $A$  is **nonsingular** (a.k.a. *invertible*). Otherwise, we'll say that  $A$  is **singular**.

## Theorem ( $2 \times 2$ case)

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . If  $ad - bc \neq 0$ , then  $A$  is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If  $ad - bc = 0$ , then  $A$  is singular.

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The quantity  $ad - bc$  is called the **determinant** of  $A$  and may be denoted in several ways

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Find the inverse if possible

(a)  $A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$   $\det(A) = 3 \cdot 5 - (-1) \cdot 2 = 17$   
 $\det(A) \neq 0$ ,  $A^{-1}$  exists  
(i.e.  $A$  is nonsingular)

$$A^{-1} = \frac{1}{17} \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5/17 & -2/17 \\ 1/17 & 3/17 \end{bmatrix}$$

(b)  $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$   $\det(A) = 3 \cdot 4 - 6 \cdot 2 = 0$   $\det(A) = 0$   
 $A^{-1}$  doesn't exist  
 $A$  is singular

## Theorem

If  $A$  is an invertible  $n \times n$  matrix, then for each  $\mathbf{b}$  in  $\mathbb{R}^n$ , the equation  $A\mathbf{x} = \mathbf{b}$  has unique solution  $\mathbf{x} = A^{-1}\mathbf{b}$ .

First if  $\vec{x} = A^{-1}\vec{b}$ . Then

$$A\vec{x} = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b}.$$

So  $A^{-1}\vec{b}$  solves the eqn.

Conversely, from  $A\vec{x} = \vec{b}$ . Mult. by  $A^{-1}$

$$A^{-1}A\vec{x} = A^{-1}\vec{b} \Rightarrow I\vec{x} = A^{-1}\vec{b}$$

so  $\vec{x} = A^{-1}\vec{b}$ .  $A^{-1}\vec{b}$  is the only solution.

## Example

Solve the system

$$3x_1 + 2x_2 = -1$$

$$-x_1 + 5x_2 = 4$$

As a matrix equation

$$\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$A \vec{x} = \vec{b}$$

$$\text{We have } A^{-1} = \frac{1}{17} \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$$

$$\text{The solution } \vec{x} = A^{-1} \vec{b} = \frac{1}{17} \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \frac{1}{17} \begin{bmatrix} -13 \\ 11 \end{bmatrix} = \begin{bmatrix} \frac{-13}{17} \\ \frac{11}{17} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} \frac{-13}{17} \\ \frac{11}{17} \end{bmatrix} \text{ i.e. } x_1 = \frac{-13}{17} \text{ and } x_2 = \frac{11}{17}$$

## Theorem

(i) If  $A$  is invertible, then  $A^{-1}$  is also invertible and

$$\left(A^{-1}\right)^{-1} = A.$$

(ii) If  $A$  and  $B$  are invertible  $n \times n$  matrices, then the product  $AB$  is also invertible<sup>1</sup> with

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(iii) If  $A$  is invertible, then so is  $A^T$ . Moreover

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

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<sup>1</sup>This can generalize to the product of  $k$  invertible matrices. 

# Elementary Matrices

**Definition:** An **elementary** matrix is a square matrix obtained from the identity by performing one elementary row operation.

Examples:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$3R_2 \rightarrow R_2$$

$$2R_1 + R_3 \rightarrow R_3$$

$$R_1 \leftrightarrow R_2$$

## Action of Elementary Matrices

Let  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ , and compute the following products

$E_1A$ ,  $E_2A$ , and  $E_3A$ .

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix}$$

$E_1$  resulted from  $3R_2 \rightarrow R_2$

It performs this on  $A$ .

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} a & b & c \\ d & e & f \\ 2a+g & 2b+h & 2c+i \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## Remarks

- ▶ Elementary row operations can be equated with matrix multiplication (multiply on the left by an elementary matrix),
- ▶ Each elementary matrix is invertible where the inverse *undoes* the row operation,
- ▶ Reduction to rref is a sequence of row operations, so it is a sequence of matrix multiplications

$$\text{rref}(A) = E_k \cdots E_2 E_1 A.$$

## Theorem

An  $n \times n$  matrix  $A$  is invertible if and only if it is row equivalent to the identity matrix  $I_n$ . Moreover, if

$$\text{rref}(A) = E_k \cdots E_2 E_1 A = I_n, \quad \text{then} \quad A = (E_k \cdots E_2 E_1)^{-1} I_n.$$

That is,

$$A^{-1} = \left[ (E_k \cdots E_2 E_1)^{-1} \right]^{-1} = E_k \cdots E_2 E_1.$$

The sequence of operations that reduces  $A$  to  $I_n$ , transforms  $I_n$  into  $A^{-1}$ .

**This last observation—operations that take  $A$  to  $I_n$  also take  $I_n$  to  $A^{-1}$ —gives us a method for computing an inverse!**

## Algorithm for finding $A^{-1}$

To find the inverse of a given matrix  $A$ :

- ▶ Form the  $n \times 2n$  augmented matrix  $[A \quad I]$ .
- ▶ Perform whatever row operations are needed to get the first  $n$  columns (the  $A$  part) to rref.
- ▶ If  $\text{rref}(A)$  is  $I$ , then  $[A \quad I]$  is row equivalent to  $[I \quad A^{-1}]$ , and the inverse  $A^{-1}$  will be the last  $n$  columns of the reduced matrix.
- ▶ If  $\text{rref}(A)$  is NOT  $I$ , then  $A$  is not invertible.

**Remarks:** We don't need to know ahead of time if  $A$  is invertible to use this algorithm.

If  $A$  is singular, we can stop as soon as it's clear that  $\text{rref}(A) \neq I$ .

## Examples: Find the Inverse if Possible

$$(a) A = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix} \quad \text{Set up } [A \ I]$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$$4R_1 + R_2 \rightarrow R_2$$

$$2R_1 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{bmatrix}$$

$$2R_2 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 10 & 2 & 1 \end{bmatrix}$$

↑  
Column 3 is not a pivot column

$A^{-1}$  doesn't exist.  $A$  is singular.

## Examples: Find the Inverse if Possible

(b)  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} = A$       set up  $[A \ I]$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$-5R_1 + R_3 \rightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{bmatrix}$$



$$4R_2 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{bmatrix}$$

$$-4R_3 + R_2 \rightarrow R_2$$

$$-3R_3 + R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 2 & 0 & 16 & -12 & -3 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{bmatrix}$$

$$-2R_2 + R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{bmatrix}$$

$A^{-1}$  exists and

$$A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

## Solve the linear system if possible

$$\begin{aligned}x_1 + 2x_2 + 3x_3 &= 3 \\x_2 + 4x_3 &= 3 \\5x_1 + 6x_2 &= 4\end{aligned}$$

as  $\vec{A}\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

$$\vec{x} = \vec{A}^{-1} \vec{b} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$